3rd problem set for Probability and Statistics — 2/4 March

Independent events

Reminder: If I is an arbitrary set of indices, the events A_i for $i \in I$ are *independent* if for every finite set $J \subseteq I$

$$P\left(\bigcap_{j \in J} A_j\right) = \prod_{j \in J} P(A_j).$$

If the condition holds only for two-element sets J, we call the events $\{A_i : i \in I\}$ *pairwise* independent.

1. (a) We will model two coin flips by the uniform probability space $\{HH, HT, TH, TT\}$. Verify that the events "first flip came up heads" and "second flip came up heads", denoted A_1 and A_2 respectively, are independent.

(b) We again have a probability space with four elementary events HH, HT, TH, and TT, but this time it is not uniform. As in the previous example, A_1 is the event "first letter is H" and A_2 is the event "second letter is H". We assume that $P(A_1) = p_1$, $P(A_2) = p_2$, and that events A_1 and A_2 are independent. Verify that this determines the probability of each event in this probability space.

2. Prove that if events A, B are independent, then so are the events A, B^c. And the same holds for events A^c, B^c.

3. (a) Is it possible for events A, B to be independent and disjoint at the same time?

(b) Is it possible for events A, B to be independent and at the same time $A \subseteq B$?

Random variables

4. Three friends decide to go swimming on some day of a given week but don't fix the day. So, each one shows up at the swimming pool on a (uniformly) random day, independently. Consider the random variable X to be the number of friends (from these three) who went on Friday. Find the probability distribution of X. Generalize this to n friends.

5. Shaq shoots a basketball at a basket, on each trial he has probability of hitting 1/10, the trials are independent. He quits after the first hit. Let X denote the total number of shots.

(a) What is $P(X > k)$? (Try first for $k = 1$, $k = 2$.)

(b) What is the distribution of X? That is, determine the probability function p_X, i.e. for each x determine $P(X = x)$ (do you know the name of this distribution?).

(c) What is $P(X \geq 10 \mid X \geq 5)$?

6. Continuing from the last problem: let's denote $Y = X \bmod 2$, i.e. $Y = 0$ if X is even, otherwise $Y = 1$. Determine the distribution of Y.

7. Beatrice also shoots a basketball, she has probability p of hitting the basket. Let Z denote the number of hits from n attempts. Determine the distribution of Z.

8. Let X and Y be discrete random variables on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $f : \mathbb{R} \to \mathbb{R}$ be a function. Prove that $f(X)$ and $X + Y$ are discrete random variables.

Bonus Problem

9. (St. Petersburg Casino) We flip a coin repeatedly. If the first time the coin comes up heads on the nth flip, we get a reward of 2^n Czech crowns. How much would you be willing to pay to participate in this game?

More practice problems

10. In an election, people vote for two candidates, A and B. When leaving the polling station voters are randomly asked to participate in an exit-poll. Assume that whoever answers will answer truthfully who they voted for, but not everyone will participate. If we denote by E the set of voters who participate in the exit-poll, then suppose $P(E \mid A) = 0.7$ and $P(E \mid A^c) = 0.4$. The exit-poll results are 60 % for A. What is the actual proportion of people who voted for A?