# 2nd problem set for Probability and Statistics — 23/25 February

**1.** Prove that if $A, B$ are events, we have $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

**2.** Roll two dice. Let $A$ be the event that the first die is 1, $B$ be the event that the second die is 1, and $C$ be the event that the sum of the two dice is 7. Show that these three events are not independent, but any two of them are.

### Conditional probability, Bayes' theorem

**3.** Does $P(A \mid B) > P(A)$ imply $P(B \mid A) > P(B)$?

**4.** If $B$ is fixed, then for all $A \in F$, define $\mathbb{P}(A) = \mathbb{P}(A|B)$, prove that $(\Omega, F, \mathbb{P}')$ is a probability space.

**5.** We have three normal 6-sided dice and one die with three 1s and three 2s. We pick one of the dice uniformly at random and roll it.
(a) What is the probability of rolling a 1?
(b) If the outcome was 1, what is the probability that we picked a normal die?

**6.** Peter gets a lot of emails, but 80 % of them are spam. His spam filter correctly flags 90 % of the spam, but it also flags 5 % of the regular emails as spam.
(a) What percentage of the emails will be marked as spam?
(b) What percentage of proper emails are among those marked as spam?
(c) What percentage of spam emails are among the emails that pass the filter?

**7.** (Monty Hall Paradox) In a TV competition, the contestants (i.e., us) stand on a stage in front of three doors. Behind one door there is a car (that's what we want), behind each of the other two doors there is a goat (we don't want that). We choose one door and once we open it, we can take whatever is behind it. However, before we open it, the presenter opens one of the other doors, shows the goat behind it, and offers us the opportunity to change our choice. Should we do that? Will it increase the probability of getting a car? Note that the assignment has (at least) the following two variants:

(a) The presenter knows where the car is, and will always pick a door with a goat to open;

(b) The presenter tosses a coin to decide which door to open (of the two that we haven't chosen). If he had revealed the car, we would have lost, but that didn't happen.

To make it easier to talk about this problem: we pick door number 1, the car is behind a random door. After the moderator opens door 2 or 3, we change our choice. Calculate the probability that we win the car, in variants (a), (b).

### Bonus problems

**8.** (Simpson's Paradox) In this problem, we will have two kinds of candy: tasty yellow candies and disgusting green candies. However, we pick the candies without looking (or we are colour-blind). The candies are in four containers: a white jar and a black jar and a red bag and a black bag. Decide whether the following strange phenomenon can happen:

- The likelihood or getting a tasty candy is greater if we take it out of the the red jar than if we take it out of the black jar.
- The likelihood or getting a tasty candy is greater if we take it out of the red bag than if we take it out of the black bag.
- Now we transfer all candies from red bag into the red jar, and all candies from the black bag into the black jar. After this operation, we are more likely to pull a tasty candy if we reach into the black jar than if we reach into the red jar.