The Elbe Sandstones Geometry Workshop

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edited by Jakub Černý

Contents

1 Preface

A workshop in discrete and computational geometry was organized in the Czech Republic again after several years. This time we decided to locate the workshop outside of Prague, in Northern Bohemia. We were quite surprised since almost everybody invited to the workshop really agreed to come. As a result, the number of participants (27) was roughly twice as much as we had originally planed. But finally everything worked nicely, in our opinion, including the weather, the accommodation in a newly renovated castle in Rynartice, and a one-day trip to the national park Czech Switzerland, where no-one got lost in spite of a complicated plan involving splitting and re-joining subgroups of participants.

The official scientific programme consisted of 15 talks and a problem session. We had the opportunity to listen to leading researchers in the field and to learn about a recent progress in solving well-known geometric questions.

Photos and other information about the workshop can be found on `http://www.ms.mff.cuni.cz/acad/kam/valtr/rynartice/`.

We would like to thank all the participants for their contribution to the nice and productive atmosphere of the workshop. We thank Robert Babilon, Jakub Černý, Helena Nyklová, Hana Polišenská, and Jan Vondrák for their help with the organization of the workshop. The workshop was financially supported by the Institute for Theoretical Computer Science (ITI, `http://iti.mff.cuni.cz/`).

Pavel Valtr and Jiří Matoušek

2 Schedule of talks

TUE	9:10 - 10:00	G. Rote: Unfolding of polygons I
	10:10 - 10:40	K. Kuperberg: Pach's animal problem
	11:10 - 12:00	M. Sharir: Lenses and their descendants I
	12:10 - 12:40	O. Cheong: The Voronoi Game
	17:00 - 17:30	Cs. Toth: BSP for line segments with few distinct directions
	17:35 - 18:05	W. Kuperberg: Ball packings in a cube, vertex-inscribed regular simplices, and Hadamard matrices
	18:15 - 18:45	M. Joswig: Projectivities, Colorings, and Branched Coverings
WED	9:10 - 10:00	G. Rote: Unfolding of polygons II
	10:10 - 10:40	G. Toth: The string graph problem
	11:10 - 12:00	M. Sharir: Lenses and their descendants II
	12:10 - 12:40	V. Koltun: Almost Tight Upper Bounds for Vertical Decompositions in Four Dimensions
	20:00 - 21:00	Problem session
FRI	9:10 - 10:00	N. Linial: Recent progress in metrical embeddings I
	10:10 - 10:40	N. Linial: Recent progress in metrical embeddings II
	11:10 - 12:00	G. Tardos: Distances and distinct sums
	12:10 - 12:40	Gy. Elekes: On the Number of Distinct Radii
	17:00 - 17:30	F. Shahrokhi: Algorithms and Covering Theorems For Pseudo-Transitive Graphs With Geometric Applications
	17:35 - 17:55	B. Aronov: Distinct distances in 3 dimensions
	17:55 - 18:15	S. Har-Peled: A replacement for Voronoi diagrams of near-linear size
	18:20 - 18:50	T. Kaiser: Line transversals to unit discs
SAT	9:10 - 10:00	S. Felsner: Combinatorial Representations of Arrangements of Pseudolines
	11:10 - 12:00	U. Wagner: On the number of corner cuts
	12:10 - 12:40	Gy. Karolyi: Erdos-Szekers theorem with forbidden order types

3 Abstracts

Distinct distances in 3 dimensions

Boris Aronov

We show that the number of distinct distances determined by any set of n points in three dimensions is at least $\Omega(n^{245/453}/2^{c\alpha^2(n)}) \approx n^{0.5408}$, for some absolute constant $c > 0$, where $\alpha(n)$ is the inverse Ackermann's function. This improves the best known bound, which is somewhat smaller than $\Omega(n^{1/2})$. We also show that there always exists a point p in the given set with $\Omega(n^{49/93}/2^{c'\alpha^2(n)}) \approx n^{0.526}$ distinct distances from p to the remaining points of the set, for another absolute constant $c' > 0$.

The Voronoi Game

Otfried Cheong

This talk was based on a joint work with Hee-Kap Ahn, Siu-Wing Cheng, Mordecai Golin and René van Oostrum

We consider a competitive facility location problem with two players. Players alternate placing points, one at a time, into the playing arena, until each of them has placed n points. The arena is then subdivided according to the nearest-neighbor rule, and the player whose points control the larger area wins. We present a winning strategy for the second player, where the arena is a circle or a line segment. We also consider a variation where players can play more than one point at a time for the circle arena.

Combinatorial Representations of Arrangements of Pseudolines and Applications

Stefan Felsner

We review several representations of arrangements of pseudolines and indicate applications of these representations.

Allowable sequences: Ungar's solution to the slope problem
Enumeration of 'at most k sets' in the plane.

Wiring diagrams: The edges of a sperical arrangement graphs can be colored red and blue such that each color class is a Hamiltonian cycle.
A planarized bijection between simple allowable sequences and standard Young tableaux of staircase shape.

Local 0,1 sequences: The number of combinatorially different arrangements of pseudolines is at most 2^{cn^2}, with $c < 0.7$.

Zonotopal tilings: The complexity of the middle level is in $O(n^{3/2})$.

Triangle orientations: The representation of arrangements by triangle orientations induces an order structure on the set of all simple arrangements of n pseudolines, this order is the higher Bruhat order $B(n, 2)$. The cover relation in $B(n, 2)$ is the triangular flip. Application, e.g., in the generation of random arrangements.

A Replacement for Voronoi Diagrams of Near Linear Size

Sariel Har-Peled

For a set P of n points in $\Re^d$, we define a new type of space decomposition. The new diagram provides an ϵ-approximation to the distance function associated with the Voronoi diagram of P, while being of near linear size, for $d \geq 2$. This contrasts with the standard Voronoi diagram that has $\Omega\left(n^{\lceil d/2 \rceil}\right)$ complexity in the worst case.

Projectivities, Coloring, and Branched Coverings

Michael Joswig

Starting from a locally finite pure simplicial complex satisfying a natural connectivity property we give an elementary combinatorial construction of a branched cover. Our main results show that this yields a rather complete description of the topological phenomena arising in dimensions up to three.

The key tool for our investigation is the group of projectivities of a simplicial complex. Originally devised for the study of certain coloring problems this group turns out to encode quite a lot information on branched covers. This is similar to the situation for unbranched coverings which are known to be described by the fundamental group. A key step in the understanding of the coverings of a sufficiently connected topological space is the construction of the universal cover, which is characterized by the property that its fundamental group vanishes. Analogously, in our context, we have the *unfolding* which is characterized by the property that its group of projectivities is trivial.

Joint work with Ivan Izmestiev.

Line transversals to unit disks

Tomáš Kaiser

Katchalski and Lewis proved in 1980 that if $\mathcal{F}$ is a finite disjoint family of translates of a compact convex set in the plane and if any 3 members of $\mathcal{F}$ a have a line transversal, then there is a line meeting all but C of the sets, where C is a constant. They proved the theorem with $C = 192\pi$ and conjectured that in fact $C = 2$ would suffice. We discuss the case when $\mathcal{F}$ consists of unit disks, establishing the result with $C = 25$.

(Update: Since the talk, I have succeeded in improving the value of C to 12, mainly thanks to several helpful suggestions from the audience. A preprint of the paper is available at `http://home.zcu.cz/~kaisert/tx/line.ps.gz`.)

Erdős–Szekeres Theorem with Forbidden Order Types

Gyula Károlyi

The classical Erdős–Szekeres theorem states that *for every integer $n \geq 3$ there is an N_0 such that, among any set of $N \geq N_0$ points in general position in the plane, there is the vertex set of a convex n-gon*. Reversing the question one may ask, for any $n \geq 3$, for the largest number $f(n)$ such that any set of n points in general position in the plane contains the vertex set of a convex $f(n)$-gon. It is known that $f(n) = \Theta(\log n)$.

The relation between the Erdős–Szekeres problem and Ramsey's theorem has a wide literature. From the viewpoint of Ramsey theory, the vertex set of a convex polygon (which we will simply refer to as a convex polygon) can be called as a *homogeneous* subset of the underlying set. The Erdős–Szekeres theorem claims that no sufficiently large set can avoid 'large' homogeneous subsets. In classical graph-Ramsey theory, homogeneous subsets of the vertex set of a graph are those sets which induce either complete or empty subgraphs, that is, cliques and independent sets. According to a conjecture of Erdős and Hajnal, for every graph H there is a positive constant $\varepsilon = \varepsilon(H)$ such that every graph on n vertices which does not contain H as an induced subgraph contains a large homogeneous set whose size is at least n^{ε}. This conjecture is verified only for certain classes of graphs, and such graphs are said to possess the *Erdős–Hajnal property*. In this talk we extend these notions in the context of the Erdős–Szekeres problem.

Throughout this paper we will always assume that every point set is in general position in the plane, that is, no three points of the configuration are collinear. Two such configurations are said to be of the same *order type* if there is a one-to-one correspondence between them which preserves the orientation of each triple. Thus, order types are equivalence classes of configurations. We will say that the configuration P contains the order type $\mathcal{T}$ if there is a subset of P which belongs to $\mathcal{T}$. Ramsey theoretic aspects of order types have already been studied by Nešetřil and Valtr.

Now let $\mathcal{T}$ be a fixed order type which is not in convex position, or more generally, let $\mathcal{T}$ be an arbitrary family of such order types. Define, for $n \geq 3$, $f_{\mathcal{T}}(n)$ as the largest number such that any set of n points in general position in the plane, which does not contain $\mathcal{T}$, contains the vertex set of a convex $f(n)$-gon. Clearly $f_{\mathcal{T}}(n) \geq f(n)$, and $f_{\mathcal{T}}$ is a monotone increasing function.

We say that $\mathcal{T}$ has the *Erdős–Hajnal property* if there exists a positive constant ε, depending only on $\mathcal{T}$, such that $f(n) > n^{\varepsilon}$. If in addition there exists a positive constant c such that $f_{\mathcal{T}}(n) > cn$, then we say that $\mathcal{T}$ possesses the *strong Erdős–Hajnal property*. The aim of this talk is to study which order types admit the strong Erdős–Hajnal property, and to see if every order type has the Erdős–Hajnal property, a question raised also by Gil Kalai.

Let $k \geq 3$ and $E = \{a, b_1, b_2, \ldots, b_k\}$ be a configuration such that $b_1 b_2 \ldots b_k$ is a convex k-gon inside the triangle ab_1b_k. Then E belongs to a unique order type that we denote by $\mathcal{E}_k$. If, for example, $\mathcal{T} = \mathcal{E}_3$, it is clear that $f_{\mathcal{T}}(n) = n$ for every $n \geq 3$.

Theorem *Let $\mathcal{T}$ be any order type whose convex hull is a triangle. $\mathcal{T}$ has the strong Erdős–Hajnal property if and only if $\mathcal{T} = \mathcal{E}_k$ for some integer $k \geq 3$.*

Corollary *If an order type $\mathcal{T}$ has the strong Erdős–Hajnal property then for every order type $\mathcal{S} \subseteq \mathcal{T}$ whose convex hull is a triangle there is an integer $k \geq 3$ such that $\mathcal{S} = \mathcal{E}_k$.*

In fact, we give a somewhat stronger necessary condition and also show that $f_{\mathcal{S}}(n) = O(\sqrt{n})$ holds for every order type $\mathcal{S}$ that does not satisfy this condition.

A class of order types which have the Erdős–Hajnal property is what we call half-moons. Let $k, \ell \geq 3$ and $F = \{a_1, a_2, \ldots, a_\ell, b_2, \ldots, b_{k-1}\}$ be a configuration such that $a_1 a_2 \ldots a_\ell$ is a convex ℓ-gon containing $b_2, \ldots, b_{k-1}$ and $a_1 b_2 \ldots b_{k-1} a_\ell$ is a convex k-gon with lines $a_1 b_2$ and $a_\ell b_{k-1}$ intersecting segments $a_\ell a_{\ell-1}$ and $a_1 a_2$, respectively. In particular, $\{a_1, a_i, a_\ell, b_2, \ldots, b_{k-1}\}$ is of type $\mathcal{E}_k$ for every $1 < i < \ell$. Then F belongs to a unique order type that we denote by $\mathcal{F}_{k,\ell}$. Thus, $\mathcal{F}_{k,3} = \mathcal{E}_k$.

Theorem *Every order type $\mathcal{F}_{k,\ell}$ $(k, \ell \geq 3)$ has the Erdős–Hajnal property.*

There are, however, order types which do not have the Erdős–Hajnal property. The following result indicates that the analogue of the Erdős–Hajnal conjecture is not true for order types in general. It may even be the case that $f_{\mathcal{T}} = f$ for the configuration $\mathcal{T}$ in the following theorem.

Theorem *There is an order type $\mathcal{T}$ such that $f_{\mathcal{T}}(n) < \log n + 2$. More precisely, for every integer $n \geq 4$ there is a configuration of 2^{n-2} points in general position in the plane which contains neither $\mathcal{T}$ nor a convex n-gon.*

Open Problem *Characterize those configurations wich possess the strong Erdős–Hajnal property. In particular, does $\mathcal{F}_{4,4}$ have this property?*

This talk is based on a joint work with József Solymosi.

Almost Tight Upper Bounds for Vertical Decompositions in Four Dimensions

Vladlen Koltun

We show that the complexity of the vertical decomposition of an arrangement of n fixed-degree algebraic surfaces or surface patches in four dimensions is $O(n^{4+\epsilon})$, for any $\epsilon > 0$. This improves the best previously known upper bound for this problem by a near-linear factor, and settles a major problem in the theory of arrangements of surfaces, open since 1989. The new bound can be extended to higher dimensions, yielding the bound $O(n^{2d-4+\epsilon})$, for any $\epsilon > 0$, on the complexity of vertical decompositions in dimensions $d \geq 4$. We also describe the immediate algorithmic applications of these results, which include improved algorithms for point location, range searching, ray shooting, robot motion planning, and some geometric optimization problems.

Pach's animal problem

Krystyna Kuperberg

An animal in R^n is the union of lattice cubes homeomorphic to the n-ball. Two animals are equivalent if one can be obtained from the other by a finite sequence of moves each consisting of either adding or removing a cube provided the result is an animal at each move. It is known that any two animals in R^2 are equivalent and the equivalence to a single square can be obtained by removing a square at each move. J. Pach asked whether any two animals in R^3 are equivalent. The first example of an animal in R^3 not equivalent to a single cube by only removing cubes was given by J. O'Rourke, so it is necessary to enlarge the animal first in order to start to remove cubes.

If A is an animal then the animal kA is defined as a homothetic copy of A by a factor of k, i.e., consisting of k^d unit cubes. Allowing the additional operation of enlarging an animal by a factor of 3, any two animals in R^3 are equivalent in the modified sense. There is an algorithm reducing $3A$ to an animal "simpler" than A. After finitely many steps of enlarging the animal and applying the algorithm one obtains a single cube.

Enlarging an animal A to obtain kA is a tedious procedure of adding identical layers. Examples show that it can not be carried out using one direction at a time parallel to an axis. It appears that an animal can be enlarged by "blowing it up" in a direction of a corner of a large cube containing the animal. This algorithm has not been fully described yet and it stands as a conjecture.

Ball packings in a cube, vertex-inscribed regular simplices, and Hadamard matrices (work joint with Greg Kuperberg)

Wlodzimierz Kuperberg

The following finite packing problem is considered: What is the maximum radius of k congruent balls that can be packed in the n-dimensional unit cube? Most of the known results in dimension 3 are due to J. Shaer (no solution is known for $n = 3$ and $k > 10$). In dimensions greater than 3 exact results are few and far between (some solved cases in dimensions $n = 4$ and $n = 5$ will be presented). S.S. Ryshkov (et al) solved the problem for $k = 3$ and all n; also they noticed a connection with Hadamard matrices: if there exists a Hadamard matrix of size $n+1$, then there is a regular n-simplex vertex-inscribed in, and concentric with, the n-cube, and then the solution of the packing problem for $k = n+1$ presents itself readily. Generalizing their observation, we consider the problem of existence of a d-dimensional regular simplex vertex-inscribed in, and concentric with, the n-cube. The necessary conditions for its existence are that $d \equiv 3 \pmod 4$ and n is a multiple of d (the Hadamard case), or $d \equiv 1 \pmod 4$ and n is an even multiple of d. This problem is equivalent to the problem of existence of certain resolvable cominatorial designs, that include the Hadamard designs. We show that for every odd d the cube of dimension $\frac{1}{2}\binom{d+1}{\frac{d+1}{2}}$ contains a concentric, vertex-inscribed d-simplex, and if a Hadamard matrix of size $2d+2$ exists, then the $2d$-cube contains such a simplex. Hence, a confirmation of the Hadamard matrix conjecture would provide a complete solution to the problem of centrally inscribing a regular d-simplex in the n-cube.

Recent advances in metrical embeddings

Nati Linial

Finite metric spaces have been the focus of intensive research in recent years. They turn out to have interesting implications in combinatorics, in the theory of algorithms and in various subfields of geometry. In this talk I gave a general overview of the present state of the field. Finally I mentioned the following recent result (joint with Magen and Naor): Let G be a k-regular graph $k > 2$ and girth g. Then every embedding of G into Euclidean space has distortion $\Omega(\sqrt{g})$.

Unfolding of polygons

Güenter Rote

Every planar polygon can be continuously deformed into convex position, without self-intersections, if the vertices and edges are considered as movable joints connected by fixed-length bars. Moreover, distances between vertices never decrease during this motion. This result was recently obtained together with Bob Connelly and Erik Demaine. The proof is not deep in itself but involves a sequence of reductions. In particular, it draws on results and concepts from rigidity theory, which I introduce and prove: duality between motions and self-stresses of frameworks, and the Maxwell-Cremona correspondence between planar self-stresses and three-dimensional polyhedral terrains. I also describe an alternative approach due to Ileana Streinu, that is more algorithmic in nature and involves so-called pseudo-triangulations of planar point sets, which are interesting in their own right.

Algorithms and Covering Theorems For Pseudo-Transitive Graphs With Geometric Applications

Farhad Shahrokhi

(This research was supported by NSF grant CCR-9988525.)

A directed acyclic graph $G = (V, E)$ $n = |V|$ and $m = |E|$, is pseudo-transitive with respect to a given subset of edges E_1, if $ab \in E_1$ and $bc \in E$ implies that $ac \in E$. When $G = (V, E)$ is pseudo-transitive with respect to E_1, we write $G = (V, E_1, E)$. It is easy to see that the complement of the intersection graph of any finite set of bounded and closed subsets of R^k has an orientation which is pseudo-transitive. In this the edge set E_1 is induced by the separability properties with respect to a class of hyperplanes.

Let $G = (V, E_1, E)$ be pseudo-transitive, and let $G_2 = (V, E_2)$, where $E_2 = E - E_1$. We give two exact algorithms for computing longest chains in pseudo transitive graphs. The first algorithm computes a longest chain of any pseudo-transitive graph G, in $O(n^{\omega_2+1}m)$ time, where ω_2 is the length of a longest chain in the graph $G_2 = (V, E_2)$. This algorithms can be applied to different classes of map labeling problems, and its time complexity is better than the square root of the previous algorithms. When E_1, and $E_2 = E - E_1$ are partial orders on V, we present a second algorithm that computes a longest chain of G in $O(\sum_{x \in V} \deg^2(x))$, time where $deg(x)$ is the degree of x.

We also derive approximate chain-antichain covering results in certain classes of pseudo-transitive graphs. The results imply that the gaps between the chromatic numbers and the largest clique sizes, and the gaps between the clique cover numbers and the independence numbers, are small, in many intersection graphs whose underlying vertices are the closed and bounded sets in R^k. In particular, for many classes of intersection graphs of subsets of R^2, $\beta = O(\alpha \log(L))$, and $\chi = O(\omega . \log(L))$, where χ and β denote chromatic number and clique cover number, and α and ω denote the sizes of a largest independent set, and a largest clique, and L is a parameter whose value is smaller or equal to α and depends on the sizes and shapes of the subsets. Our general results also imply that gaps between the traversal numbers and largest independent sets of some of the interesting classes of subsets of R^2 are small.

Lenses and Their Descendants

Micha Sharir

We present recent progress in the study of incidences between points and circles in the plane, and many related problems. The main improved bound for the number of incidences between m points and n general circles in the plane, due to Aronov and Sharir, is

$$O(m^{2/3}n^{2/3} + m^{6/11+3\epsilon}n^{9/11-\epsilon} + m + n),$$

for any $\epsilon > 0$.

The first step towards obtaining this bound is to cut the circles into pseudosegments, and then apply Székely's technique for the incidence bound. Aronov and Sharir show that $O(n^{3/2+\epsilon})$ cuts suffice to cut the circles into pseudosegments. These cuts eliminate *lenses*—a pair of arcs of different circles that have common endpoints.

A considerable portion of the new research deals with lenses. In the case of pairwise-intersecting pseudocircles, an ingenious construction of Pinchasi shows that the number of *empty* lenses (lenses not crossed by any other curve) is linear. Many other results involving lenses in such arrangements have been obtained. (A paper by Agarwal et al., in preparation, summarizes these results.)

We also present several related results:

- An effective and efficient duality transform between points and pseudolines, which has several algorithmic applications. (Work by Agarwal and Sharir and by Smorodinsky and Sharir.)
- Improved bounds for the complexity of many faces in arrangements of circles. (Work by Agarwal, Aronov and Sharir.)
- A new lower bound on the number of distinct distances in three dimensions. (Aronov and Sharir)
- New bounds for incidences between points and lines or circles in three dimensions. (Work by Sharir and Welzl, and by Aronov, Koltun and Sharir.)

Distinct sums and distinct distances

Gábor Tardos

The talk started with a short overview of the result of J. Solymosi and Cs. Toth on the Erdos problem of finding the minimal number of distinct distances n points can determine in the plain. We then destilled the following combinatorial problem: For n pairwise disjoint s element sets of reals, what is the minimum number of distinct sums we can form by adding two distinct numbers of the same set. A lower bound for this latter problem of $\Omega(n^{1/e-\epsilon})$ was presented yielding a lower bound of $\Omega(n^{4e/(5e-1)-\epsilon})$ for the original Erdos problem.

BSP with few distinct directions

Csaba Tóth

We consider binary space partitions (BSP) for n disjoint line segments in the plane. It is known that there is a BSP of size at most $O(n \log n)$, in general, and the smallest BSP can be as big as $\Omega(n \log n / \log \log n)$ in the worst case. It is shown that there exists a BSP of size $O(kn)$ if the line segments have at most k different orientations. No linear upper bound was known, so far, for any fixed $k > 2$.

It is shown that there is an autopartition such that each line segment is cut at most $O(k)$ times. The key tools of the proof are cycles and convex cycles defined on line segments touching the convex hull.

String representations of graphs

Géza Tóth

A graph is called a *string graph* if its vertices can be represented by continuous curves ("strings") in the plane so that two of them cross each other if and only if the corresponding vertices are adjacent.

It was shown by Kratochvíl that the problem of recognizing string graphs is NP-hard. Ten years ago Kratochvíl and Matoušek exhibited a string graph G on n vertices such that in every representation of G there are at least 2^{cn} crossings. They raised the question whether it is *decidable* that a graph is a string graph. We answer this question in the affirmative by finding a a recursive function $f(n)$ with the property that every string graph has a representation with at most $f(n)$ crossings. We will also discuss some related problems on crossing numbers. (Joint work with János Pach.)

On the Number of Corner Cuts

Uli Wagner

A *corner cut* in dimension d is a finite subset T of $\mathbf{N}_0^d$ that can be separated from its complement by an affine hyperplane disjoint from $\mathbf{N}_0^d$. Corner cuts were first investigated by Onn and Sturmfels [2]. Their motivation stems from computational commutative algebra; for instance, they show that if $I \subset K[x_1, \ldots, x_d]$ is the vanishing ideal of a generic configuration of k points in K^d, K any infinite field, then the corner cuts of fixed size k in dimension d are in one-to-one correspondence with the (reduced) Gröbner bases of I.

Let us write $c_d(k)$ for the number of corner cuts of cardinality k in d dimensions. Apart from the above-mentioned relations to algebraic geometry, estimating the number $c_d(k)$ of such corner cuts seems to be of interest in its own right, since it is a rather natural special instance of the *k-set problem*, which concerns the maximal possible number of k-element subsets T of a set S of n points in $\mathbf{R}^d$ such that T can be separated from $S \setminus T$ by a hyperplane (see [4] or [3] for recent developments).

Onn and Sturmfels give an upper bound of $O(k^{2d\frac{d-1}{d+1}})$ for $c_d(k)$ when d is fixed. Moreover, in the planar case, it is known (see [1]) that $c_2(k) = \Theta(k \log k)$. We show[1] that in general, for any fixed dimension d, the order of magnitude of $c_d(k)$ is between $k^{d-1} \log k$ and $(k \log k)^{d-1}$. In fact, the corner cuts of size k correspond to the vertices of a certain polyhedron $P_k^d \subset \mathbf{R}^d$, and our proof of the upper bound immediately extends to the overall number of flags (i.e., ascending chains $F_1 \subsetneq \ldots \subsetneq F_s$ of faces) of that polyhedron.

References

[1] Sylvie Crteel, Gaël Rémond, Gilles Schaeffer and Hugh Thomas. The number of plane corner cuts. *Advances in Applied Mathematics,* 23(1):49-53, 1999

[2] Schmuel Onn and Bernd Sturmfels. Cutting corners. *Advances in Applied Mathematics,* 23(1):29-48, 1999

[3] Micha Sharir, Shakhar Smorodinsky and Gábor Tardos. An improved bound for k-sets in three dimensions. In *Proceedings of the Sixteenth Annual Symposium on Computational Geometry,*2000

[4] Géza Tóth. Point sets with many k-sets. In *Proceedings of the Sixteenth Annual Symposium on Computational Geometry,*2000

[5] Uli Wagner. On the number of corner cuts. *Advances in Applied Mathematics,* to appear, 2001

[1] It has been communicated to me that the same bounds have been found independently by Gaël Rémond.

4 Open problems

Does the Dual Graph of a Cubical Polytope Determine its Combinatorial Type?

Michael Joswig

In general, it may happen that many convex polytopes with distinct combinatorial types (and even distinct dimensions) have isomorphic vertex-edge graphs. The most prominent examples are the cyclic polytopes in dimensions 4 and above on n vertices, which all have the complete graph K_n as their graph. At the same time this is also the graph of the $(n-1)$-dimensional simplex. Observe that the cyclic polytopes are simplicial, that is, each proper face is a simplex.

However, there are special classes of polytopes for which it is known that the graph does determine the combinatorial type. In particular, this is true for simple polytopes (that is, the duals of simplicial polytopes)by a theorem of Blind and Mani [3]. A very short and elegant proof of Kalai [6] links this result to central aspects of the combinatorics of simple polytopes.

A polytope is called *cubical* if each proper face is combinatorially equivalent to a cube. They are somewhat similar to simplicial polytopes in the following sense. Simpliciality for polytopes requires that the vertices on each facet are in general position. Now, cubicality also encodes general position phenomena: A hyperplane arragement is in general position if and only if its associated zonotope is cubical. Moreoever, on a different account, (the boundary of) a cubical d-polytope, or more generally, a cubical $(d-1)$-sphere, gives rise to a normal crossing immersion of some (not necessarily connected) hypersurface into the $(d-1)$-sphere.

As for simplicial polytopes there is no hope for cubical polytopes to be determined by their graphs. For instance, there are cubical analogues to the cyclic polytopes [5]. But, as simplicial polytopes are determined by their dual graphs (which are precisely the graphs of simple polytopes), this raises the question asked in the title.

The vertex figure of a polytope which is dual to a cubical d-polytope is always a $(d-1)$-dimensional cross polytope. Note that in a simple polytope each vertex figure is a simplex. One can analyze Kalai's proof of the Blind-and-Mani-Theorem in order to extract what can be applied to other classes of polytopes [4]. This yields that, in order to reconstruct a cubical polytope from its dual graph, it suffices to find the pairs of opposite vertices in all the cross polytopes arising as the dual vertex figures.

It is known [4] that for the very restricted class of stacked cubical polytopes the dual graph does, in fact, determine the combinatorial type. Moreover, by a result of Babson, Finschi, and Fukuda [1], the cubical zonotopes are determined by their dual graphs.

In addition to the results mentioned above Björner, Edelmann, and Ziegler [2] prove that the graph of an arbitrary zonotope also determines the combinatorial type of the zonotope. But, their proof employs entirely different techniques. As it seems the result on zonotopes does not fit into the context set by the Blind-and-Mani-Theorem.

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Drawings of Q_n

Pavel Valtr

Is it possible to draw the graph of the n-dimensional cube Q_n (having 2^n vertices and $n2^{n-1}$ edges) in the plane so that vertices are represented by points and edges by segments (resp. by Jordan curves) so that there are no 100 (say) pairwise crossing edges? (Two edges meeting in a vertex are not considered as crossing edges.) This is trivially true for small values of n but the question is to find the drawing for any n. For large n, Q_n is not planar and therefore we cannot replace 100 by 2. It is also known that we cannot replace 100 by 3 (this follows from a result of [P.K. Agarwal, B. Aronov, J. Pach, R. Pollack, M. Sharir: Quasi-planar graphs have a linear number of edges, *Combinatorica* 17 (1997), no. 1, 1–9]).

On the other hand, it is known that every drawing of a graph with n vertices and at least $c_k n \log n$ edges has k pairwise crossing edges. The graph Q_n is a candidate to show that the bound $c_k n \log n$ cannot be improved (up to the value of c_k), which motivates the above problem.

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