Strictly convex norms allowing many unit distances and related touching questions

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Abstract

We construct a (simply defined) strictly convex norm $||.||$ in the plane such that for each n there is a set of n points determining $\Omega(n^{4/3})$ unit distances with respect to $||.||$. Our construction has implications for geometric touching problems. We construct an infinite sequence $C_1, C_2, C_3, \ldots$ of (possibly overlapping) translates of the same strictly convex compact centrally symmetric body in the plane such that for each $n \geq 2$, there are $\Omega(n^{4/3})$ touching pairs among $C_1, C_2, \ldots, C_n$. This is asymptotically best possible, since there are at most $O(n^{4/3})$ touching pairs in any collection of n (possibly overlapping) translates of the same strictly convex compact body in the plane. We give connections of the above results to problems on osculation vertices and empty lenses in arrangements of pseudo-circles. We further show that the number of osculating (touching) vertices in an arrangement of n pseudo-circles is at most $O(n^{3/2} \log n)$.

1 Introduction

1.1 Unit distances

In 1946 Erdős asked what is the minimum number, $u(n)$, of unit segments (unit distances) determined by a set of n points in the Euclidean plane. The problem motivated a lot of research, e.g. see [4] for a survey. Currently the best known bounds on $u(n)$ are:

$$\Omega(n^{1+c/\log\log n}) \leq u(n) \leq O(n^{4/3}).$$

The lower bound was proved already in the original paper of Erdős [6]. It is attained by a properly scaled square lattice $\lfloor\sqrt{n}\rfloor \times \lfloor\sqrt{n}\rfloor$. Erdős' conjecture asserts that $u(n) \leq O(n^{1+c'/\log\log n})$. The original upper bound $O(n^{3/2})$ of Erdős was improved several times (e.g. see [4] for details). There are several proofs of the current upper bound, the simplest and most elegant one is due to Székely [12]. In this paper we consider the same problem in a 2-dimensional Minkowski space $(\mathbf{R}^2, ||.||)$, i.e., in the plane with some norm $||.||$. A norm $||.||$ is determined by its *unit disk* $D_{||.||} = \{p \in \mathbf{R}^2 : ||p|| \leq 1\}$, which may be any convex compact body having a non-empty interior and symmetric with respect to the origin. *The unit circle* $C_{||.||} = \{p \in \mathbf{R}^2 : ||p|| = 1\}$ is the boundary of $D_{||.||}$.

Let $u_{||.||}(n)$ be the maximum number of unit segments determined by n points in $(\mathbf{R}^2, ||.||)$, where a segment ab is *unit* if $||b - a|| = 1$.

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A norm $||.||$ in the plane is *strictly convex*, if the unit disk is strictly convex (i.e., the unit circle does not contain a straight-line segment of positive length). If a norm is not strictly convex then the unit circle contains a straight-line segment s of positive length and we can place n points on two short segments parallel to s in such a way that there will be at least $\approx (n/2)^2$ unit segments determined by the n points and thus $u_{||.||}(n) = \Theta(n^2)$. For this case the exact values of $u_{||.||}(n)$ were given by Brass [2].

If $||.||$ is strictly convex then the situation is essentially different. Székely's method [12] (as well as other proofs) gives $u_{||.||}(n) = O(n^{4/3})$ as in the Euclidean case. Brass [2] conjectured that $u_{||.||}(n)$ is almost linear for any strictly convex norm:

Conjecture 1 (Brass 1996) *For any strictly convex norm $||.||$ in $\mathbf{R}^2$ and for any $\varepsilon > 0$,*

$$u_{||.||}(n) = o(n^{1+\varepsilon}).$$

Two years later Brass [3] showed that his conjecture can hold only if the constant involved in the o-notation depends on the norm $||.||$:

Theorem 1 (Brass 1998) *There is a $c > 0$ such that for any n there is a strictly convex norm $||.||_n$ in $\mathbf{R}^2$ such that*

$$u_{||.||_n}(n) \geq cn^{4/3}.$$

In this paper we disprove Conjecture 1 in a quite strong sense:

Theorem 2 *There is a (simply defined) strictly convex norm $||.||$ in $\mathbf{R}^2$ such that*

$$u_{||.||}(n) = \Theta(n^{4/3}).$$

1.2 Touching bodies and touching curves

The number $u_{||.||}(n)$ is equal to the maximum number of touching pairs[1] in a collection of n (possibly overlapping) translates of $D_{||.||}$, since the unit segments determined by a set P of n points in $(\mathbf{R}^2, ||.||)$ are in one–to–one correspondence with the touching pairs in the collection of the n translates of $D_{||.||}$ with centers in the points of $2P$ (= the set P scaled by the factor of 2).

Therefore the following theorem is a reformulation and (slight) strengthening of Theorem 2:

Theorem 3 *(i) There is an infinite sequence $C_1, C_2, C_3, \ldots$ of (possibly overlapping) translates of the same strictly convex compact centrally symmetric body in the plane such that for each $n \geq 2$, there are $\Omega(n^{4/3})$ touching pairs among $C_1, C_2, \ldots, C_n$.*

(ii) There are at most $O(n^{4/3})$ touching pairs in any collection of n (possibly overlapping) translates of the same strictly convex compact body in the plane.

The situation is less clear if the bodies are not necessarily translates of the same centrally symmetric body. We say that n closed Jordan curves in the plane form an *arrangement of pseudo-circles*, if any two of them touch each other at most once and have at most two points in common. If a point is the only common (touching) point of two of the pseudo-circles, then it is called an *osculation vertex*. Let $p(n)$ be the maximum number of osculation vertices in an arrangement of n pseudo-circles. Erdős and Grünbaum [7] proved $\Omega(n^{4/3}) \leq p(n) \leq O(n^{5/3})$. The lower bound $p(n) = \Omega(n^{4/3})$ also follows quite easily from Theorem 3(i). As noticed by

[1]A pair of convex compact bodies $\{S, T\}$ is a *touching pair*, if $S \cap T \neq \emptyset$ and Int $S \cap$ Int $T = \emptyset$.

Micha Sharir (personal communication), $p(n)$ is closely related to the maximum number of so-called *empty lenses* in an arrangement of n pseudo-circles. A result on so-called *non-overlapping lenses* [11] can be used to derive the following upper bound on $p(n)$:

Theorem 4 $p(n) = O(n^{3/2} \log n)$.

In Section 3 we give a relatively simple direct proof of Theorem 4 based on the methods and results of [11].

We remark that analogues of Theorem 3 for collections of pairwise non-overlapping translates of a convex compact body are frequently studied (e.g., see [1] and the references in it), in particular in connection with the closely related *kissing-number (Newton-number) problem* (e.g., see [14] for the description of the problem and for further references). However, we are not aware of any previous result for translates which may arbitrarily overlap. In [10], overlappings are allowed but the intersection of any three bodies is required to be empty.

1.3 Results in three dimensions

In an earlier version of this paper we showed that $u_{||.||}(n) = \Omega(n^{3/2})$ holds for the strictly convex norm $||.||$ in $\mathbf{R}^3$ defined by $||(x,y,z)|| = \frac{\sqrt{4(x^2+y^2)+z^2}+|z|}{2}$ and having the unit sphere $S_{||.||} = \{(x,y,z) \in \mathbf{R}^3 : |z| = 1 - x^2 - y^2\}$. Peter Brass (personal communication) improved this bound to $u_{||.||}(n) = \Theta(n^2)$ for the same norm $||.||$.

Clarkson *et al.* [5] proved an upper bound $u_{||.||}(n) = O(n^{3/2}\beta(n))$ for the *Euclidean norm* in $\mathbf{R}^3$, where $\beta(n)$ is an extremely slowly growing unbounded function (more precisely $\beta(n) = 2^{O(\alpha^2(n))}$, where $\alpha(n)$ is the inverse of the Ackermann function).

2 Unit distances in the Minkowski plane

Here we present two constructions giving the lower bound in Theorem 2. The norm in the first construction is simply defined from the algebraic point of view (the unit circle is formed by two parabolic arcs). The second construction actually gives a somewhat weaker bound (weaker by a logarithmic factor) but has some interesting properties. In particular, in this construction we have some freedom in the choice of the norm. It shows that the exponent 4/3 in Theorem 2 is not attained from below by some number–theoretic "coincidence" which would be connected with the particular norm used in the first construction. The norm in the second construction gives "many" unit distances for the (properly scaled) square lattices. Consequences of this fact related to the so–called Jarník's curve [9] are elaborated in [13].

As noticed in [4], $u_{||.||}(n) = \Omega(n \log n)$ holds for any norm $||.||$. For $n = 2^k$, this bound is attained by the n-point set $\{o + \sum_{i \in S} u_i : S \subseteq \{1, 2, \dots, k\}\}$, where $u_1, \dots, u_k$ are any k "generic" unit vectors. However, this bound is far from the upper bound.

Open problem: *Find a norm* $||.||$ *in the plane with* $u_{||.||}(n) = o(n^{4/3})$.

2.1 First construction

Here we construct a strictly convex norm $||.||$ in the plane with $u_{||.||}(n) = \Omega(n^{4/3})$. This gives Theorem 2, since Székely's method [12] gives $u_{||.||}(n) = O(n^{4/3})$ for any strictly convex norm $||.||$ in the plane.

The norm is defined by

$$||(x,y)|| = \frac{\sqrt{4x^2+y^2}+|y|}{2}.$$

The unit circle $C_{||.||}$ consists of points (x,y) satisfying $|y| = 1-x^2$. It is a union of two parabolic arcs (see Fig. 1).

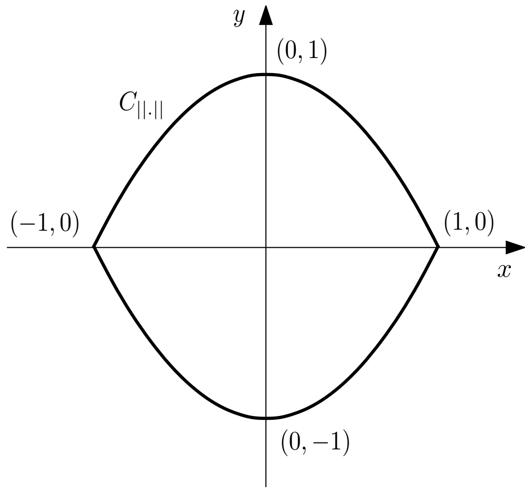

Figure 1: The unit circle of a strictly convex norm allowing many unit distances.

Let $n \geq 9$. We take the largest integer k such that $(2k+1)(2k^2+1) \leq n$. Clearly, $k = \Theta(n^{1/3})$. Set

$$P = P_k := \left\{ \left(\frac{i}{k}, \frac{j}{k^2}\right) \;:\; i \in \{-k, -k+1, \ldots, k\},\; j \in \{-k^2, -k^2+1, \ldots, k^2\} \right\}$$

(see Fig. 2).

The size of P is $(2k+1)(2k^2+1) \leq n$. We now show that it determines $\Omega(k^4) = \Omega(n^{4/3})$ unit distances with respect to $||.||$. For $i = 0, 1, \ldots, k$, let u_i be the vector connecting the origin $(0,0)$ with the point $(\frac{i}{k}, 1-\frac{i^2}{k^2})$ lying on $C_{||.||}$. See Fig. 2. Each of the $k+1$ vectors u_i has unit length (with respect to $||.||$) and can be translated to at least $(k+1)(k^2+1)$ different positions in which it connects pairs of points of P at distance 1 (indeed, any translation of u_i which translates the origin to a point of P in the third quadrant is suitable — see Fig. 2). This gives us $(k+1) \cdot (k+1)(k^2+1) = \Theta(n^{4/3})$ different unit segments determined by P.

2.2 Second construction

Here we construct a strictly convex norm $||.||$ in the plane such that, in $(\mathbf{R}^2, ||.||)$, the number of unit distances determined by the (properly scaled) square lattice $\lfloor\sqrt{n}\rfloor \times \lfloor\sqrt{n}\rfloor$ is $\Omega(n^{4/3}/\log^{1+\varepsilon} n)$, for any fixed $\varepsilon > 0$.

Fix $\varepsilon > 0$.

Let U be the upper half of the Euclidean unit circle. We use an infinite monotone sequence $(1,0) = p_1, p_2, p_3, \ldots$ of points on U partitioning U into arcs $C_1, C_2, \ldots$ such that each C_i is the (closed) subarc of U between p_i and p_{i+1} (see Fig. 3). The length of C_i is $\alpha_i := c_0/i^{1+\varepsilon}$, where c_0 is chosen so that $\sum_{i=1}^{\infty} \alpha_i = \pi$.

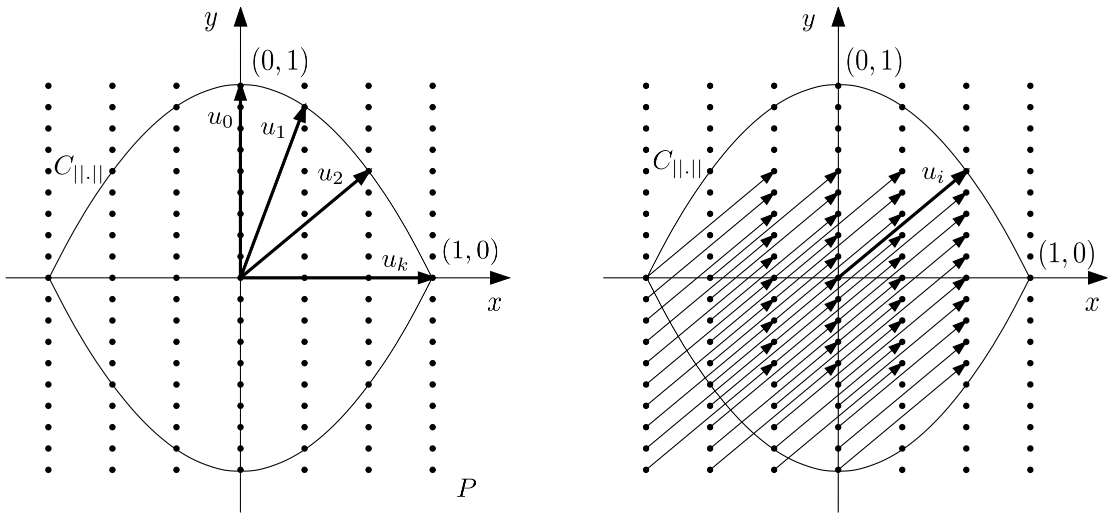

Figure 2: The point set P and the vectors u_i (on the left, case $k = 3$). The considered $(k+1)(k^2+1)$ translations of a vector u_i (on the right).

For each i let $A_i := \text{conv}(C_i)$, see Fig. 4. Our strictly convex norm $||.||$ will have the upper half of the unit circle $C_{||.||}$ contained in the union of the regions A_i. The lower half of $C_{||.||}$ will be taken symmetric to the upper half with respect to the origin.

Let $i \geq 1$. To define the part of $C_{||.||}$ lying in A_i, we use the square lattice $(2 \cdot 2^i + 1) \times (2 \cdot 2^i + 1)$ scaled to fit in the square $[-1,1] \times [-1,1]$, i.e. the lattice

$$L_i = \left\{ (\frac{j}{2^i}, \frac{k}{2^i}) : j, k \in \{-2^i, -2^i+1, \ldots, 2^i-1, 2^i\} \right\}.$$

For simplicity we suppose that the point p_i lies in the lattice L_i (otherwise the argument works after simple technical changes). Let q_i be the middle point of the arc C_i and let r_i be the center of the segment $p_i p_{i+1}$ (see Fig. 4). Next, we find a relatively large set S_i of points of L_i in the triangle $\triangle p_i q_i r_i$ such that $p_i \in S_i$ and $S_i \cup \{p_{i+1}\}$ is in convex position. This will ensure that $\bigcup_{i=1}^{\infty} S_i$ is also in convex position.

We determine S_i by choosing the slopes and lengths of vectors connecting consecutive points of S_i. For $c > 0$, the set $V_i = V_i(c)$ of these vectors will be the set of vectors $v = \left(\frac{j}{2^i}, \frac{k}{2^i}\right)$ with the following three properties:

(P1) j, k are relatively prime integers,

(P2) $p_i + v$ lies in $\triangle p_i q_i r_i$, and

(P3) $|v| \leq c 2^{-2i/3}$.

The angle $q_i p_i r_i$ has size $\Theta(\alpha_i)$. Consequently, it follows from the well-known facts about the distribution of relatively prime integer vectors that the set V_i has size $\Theta(\alpha_i \left(c 2^{-2i/3}\right)^2 2^{2i}) = \Theta(c^2 \alpha_i 2^{2i/3})$. The sum of the lengths of the vectors in V_i is therefore $O(c^2 \alpha_i 2^{2i/3} \cdot c 2^{-2i/3}) = O(c^3 \alpha_i)$, which is smaller than $|p_i - r_i| = \Theta(\alpha_i)$ for a sufficiently small constant $c > 0$ (independent of i). From now on, such a small $c > 0$ is fixed.

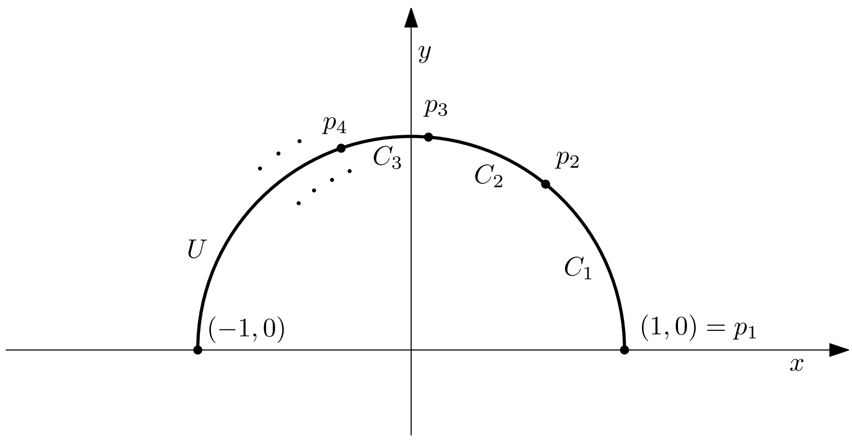

Figure 3: The points $p_1, p_2, p_3, \ldots$ partition U into arcs $C_1, C_2, \ldots$.

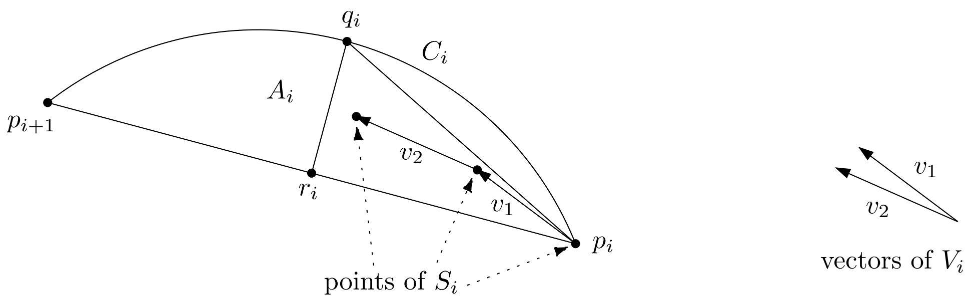

Figure 4: The region A_i, the vectors of V_i, and the points of S_i.

Due to property (P1), the vectors in V_i have pairwise different directions. Let $v_1, v_2, \ldots, v_h$ be the vectors of V_i ordered according to their directions in the counterclockwise order. We choose the set S_i as the set of the $h + 1 = |V_i| + 1$ points $p_i + \sum_{s=1}^{t} v_s$, $t = 0, 1, \ldots, h$. The points of S_i form a convex "chain" in $\triangle p_i q_i r_i$ so that the set $S_i \cup \{p_{i+1}\}$ is in convex position. The size of S_i is $|V_i| + 1 = \Theta(c^2 \alpha_i 2^{2i/3})$.

For the upper half of $C_{||.||}$ we take any strictly convex curve connecting $p_1 = (1, 0)$ with $(-1, 0)$ and containing all points of $S_1 \cup S_2 \cup \cdots$. The lower half of $C_{||.||}$ is taken symmetric to the upper half with respect to the origin. This defines the norm $||.||$.

Now, let $n \geq 4$. We choose i such that $|L_i| \leq n < |L_{i+1}| = \Theta(2^{2i})$. We now prove that L_i determines at least $\Omega(n^{4/3} / \log^{1+\varepsilon} n)$ unit segments with respect to $||.||$. Each of the $|S_i|$ unit vectors $\overrightarrow{os}, s \in S_i$, may be translated to $\Theta(|L_i|) = \Theta(n)$ positions in which it connects two points of L_i (indeed, any translation of $\overrightarrow{os}$ which translates o to a point of L_i in the third quadrant is suitable). This gives us at least $|S_i| \cdot \Theta(n) = \Theta(c^2 \alpha_i 2^{2i/3} \cdot n) = \Theta(n^{4/3} / \log^{1+\varepsilon} n)$ unit segments determined by L_i.

3 Touching bodies and touching curves

Here we prove Theorems 3 and 4.

Proof of Theorem 3. For the proof of part (i), consider the norm $||.||$ and the sets $P_k, k \geq 1$, defined in Paragraph 2.1. Place the points of $P_1 \cup P_2 \cup \cdots$ in one sequence $p_1, p_2, p_3, \ldots$ such that the points of P_1 are followed by the points of $P_2 \setminus P_1$, which are followed by the points of $P_3 \setminus (P_1 \cup P_2)$, etc. Moreover, suppose that $p_1 = (0,0)$, $p_2 = (0,1)$ (say).

For $i \geq 1$, let $C_i := 2p_i + D_{||.||}$. Thus, C_i is a translate of $D_{||.||}$ centered in the point $2p_i$. It is easily verified that the sequence $C_1, C_2, C_3, \ldots$ satisfies part (i) of the theorem.

For the proof of part (ii), we need to bound the number of touching pairs in any collection of n translates of an arbitrary strictly convex compact body C. Denote the translates by $C + u_1, \ldots, C + u_n$.

Lemma 1 *$C + u_i$ touches $C + u_j$ if and only if u_j lies on the boundary of $C' + u_i$, where $C' = C - C$.*

Proof. Suppose first that $C + u_i$ touches $C + u_j$ at a point p. Let l be a line through p separating the interiors of the bodies $C + u_i, C + u_j$ (see Fig. 5). Since

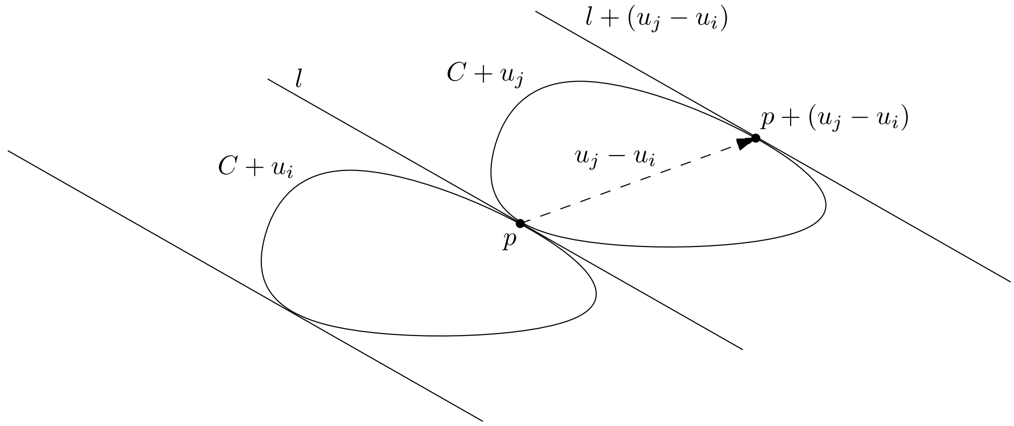

Figure 5: A line l separates the bodies $C + u_i, C + u_j$.

$p \in (C + u_i)$, the point $p' = p + (u_j - u_i)$ lies in $(C + u_i) + (u_j - u_i) = C + u_j$. It follows that $u_j - u_i = p' - p$ lies in $C' = (C + u_j) - (C + u_j)$. The body $C + u_j$ lies entirely in the closed strip bounded by the lines l and $l + (u_j - u_i)$. Therefore $C' = (C + u_j) - (C + u_j)$ does not contain $c(u_j - u_i)$ for any $c > 1$. Thus $u_j - u_i$ lies on the boundary of C'. It follows that u_j lies on the boundary of $C' + u_i$.

Suppose now that u_j lies on the boundary of $C' + u_i$. Then $u_j - u_i$ lies on the boundary of $C' = (C + u_j) - (C + u_j)$. It follows that the body $C + u_j$ contains two (boundary) points p, p' such that $p' - p = u_j - u_i$. If $p + v, p' + v$ were interior points of $C + u_j$ for some vector v, then $u_j - u_i = (p' + v) - (p + v)$ would lie in the interior of $C' = (C + u_j) - (C + u_j)$. Thus no such vector v exists. It follows that there exist two parallel lines $l, l' = l + (u_j - u_i)$ such that $p \in l, p' \in l'$, and $C + u_j$ lies entirely in the closed strip bounded by the lines l, l'. Thus $C + u_i = (C + u_j) - (u_j - u_i)$ and $C + u_j$ are separated from each other by the line l. Consequently $C + u_i$ touches $C + u_j$ at the point p. □

We now bound the number N of pairs $\{i, j\}$ such that u_j lies on the boundary of $C' + u_i$. Since $C' = C + (-C)$ is centrally symmetric and strictly convex, the boundaries of the sets $C' + u_1, \ldots, C' + u_n$ form a collection of n pseudo-circles. Székely's method [12] gives the upper bound $O(n^{4/3})$ on the number of incidences between a collection of n pseudo-circles and a set of n points. Thus $N = O(n^{4/3})$. The upper bound in Theorem 3(ii) now follows from Lemma 1. □

Proof of Theorem 4. Let $\{C_1, \ldots, C_n\}$ be a collection of n pseudo-circles. For each $i \in \{1, \ldots, n\}$, let S_i^1 be the circular sequence of the curves C_j touching the curve C_i from outside, appearing in this sequence according to the clockwise order of their touchings along the boundary of C_i. Similarly, let S_i^2 be the circular sequence of the curves C_j touching the curve C_i from inside, appearing in this sequence according to the counter-clockwise order of their touchings along the boundary of C_i.

We now show that two distinct sequences S_i^α, S_j^β do not contain the same 3-element circular subsequence. Suppose to the contrary that (C_a, C_b, C_c) be a common circular subsequence of S_i^α, S_j^β. If the pseudo-circles C_i, C_j interesect then they divide the plane into four regions and the curves C_a, C_b, C_c all lie entirely in one of these regions. Clearly, we may therefore suppose that C_i, C_j do not intersect. The curves C_a, C_b, C_c then lie in the (unique) plane region with the boundary $C_i \cup C_j$. It is not difficult to check that if C_a, C_b intersect in at most two points (as supposed) then C_c must intersect one of the curves C_a, C_b at least four times — a contradiction.

Thus, $S_i^\alpha, i = 1, \ldots, n, \alpha = 1, 2$, are $2n$ circular sequences over n elements $C_1, \ldots, C_n$ such that no pair of them contains a common 3-element circular subsequence. According to the results in [11], the total size of the sequences S_i^α is at most $O(n^{3/2} \log n)$. Theorem 4 follows. □

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