SPARSE TRIPLE SYSTEMS VIA THE DELETION METHOD

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Definition 1. A 3-uniform hypergraph (3-*graph* for short) is a tuple of sets $G = (V, E)$, where $V \subseteq \binom{V}{3}$.

Definition 2. A (v, e)-*configuration* is a 3-graph with e edges and at most v vertices.

Definition 3. A 3-graph is called (v, e)-*free* if it does not contain a (v, e)-configuration as a subgraph.

In other words, G is (v, e)-free if any set of its e edges spans more than v vertices.

Theorem 4. *For every integer $k \geq 2$ there exists $c = c(k) > 0$ such that for every integer $n \geq 3$ there exists a $(k+2, k)$-free 3-graph $G = (V, E)$ with $|V| = n$ and $|E| \geq cn^2$.*

Proof. We may assume n to be large whenever we need it. The statement for small n then follows by setting c to be sufficiently small.

Take $p = \epsilon n^{-1}$, where $\epsilon = \epsilon(k) > 0$ is to be specified. Consider a random 3-graph G on n vertices, where every edge is present with probability p, independently (this is the 3-graph analogue of the Erdős-Rényi random graph). Let the random variable X count the edges in G. Then, by linearity of expectation,

$$\mathbb{E}(X) = p\binom{n}{3}.$$

Let Y be the random variable counting $(k+2, k)$-configurations in G. On a fixed set of $k+2$ vertices there are $d = d(k)$ possible $(k+2, k)$-configurations. Each of them is present in G with probability p^k. We therefore obtain

$$\mathbb{E}(Y) \leq d\binom{n}{k+2}p^k.$$

Note that the right hand side above is indeed an overcount, since a $(k+2, k)$-configuration can span strictly fewer than $k+2$ vertices.

By linearity of expectation, we deduce

$$\mathbb{E}(X - Y) \geq p\binom{n}{3} - d\binom{n}{k+2}p^k \geq \frac{pn^3}{7} - dn^{k+2}p^k = n^2(\epsilon/7 - d\epsilon^k).$$

Choosing $\epsilon = d^{-2}$ yields

$$\mathbb{E}(X - Y) \geq \frac{n^2}{8d}.$$

So, let us set $c = (8d)^{-1}$.

Our bound on $\mathbb{E}(X - Y)$ implies the existence of a 3-graph G on n vertices in which the number of edges exceeds the number of $(k+2, k)$-configurations by at least cn^2. This means, deleting one edge from every $(k + 2, k)$-configuration in G will result in a $(k + 2, k)$-free 3-graph with n vertices and at least cn^2 egdes. □

A minor enhancement of the above argument readily gives (can you see how?) a *monotone* version of the theorem statement, claiming that G is $(\ell + 2, \ell)$-free for all $2 \leq \ell \leq k$. In particular, G will be $(4, 2)$-free, meaning no two edges will share a pair of vertices. Such 3-graphs are called *linear*, and the maximum possible size of a linear 3-graph on n vertices is $\frac{1}{3}\binom{n}{2}$ (realized by Steiner Triple Systems). Thus, for every $k \geq 3$ there exist linear 3-graphs which are 'dense', in the sense of having a positive proportion of possible edges, yet 'sparse' in the sense of being $(k + 2, k)$-free.

Asking the natural next question regarding $(k + 3, k)$-configurations leads to one of the central problems in extremal hypergraph theory, the notoriously difficult *Brown-Erdős-Sós conjecture* from 1973.

Conjecture 5 (Brown-Erdős-Sós). *For every $c > 0$ and integer $k \geq 3$ there exists $n_0 = n_0(c, k)$ such that for all integers $n \geq n_0$ any 3-graph $G = (V, E)$ with $|V| = n$ and $E \geq cn^2$ contains a $(k + 3, k)$-configuration.*

The case $k = 3$ was settled by Ruzsa and Szemereédi in 1978, and became known as the $(6, 3)$-theorem. This was one of the first applications of Szemerédi's regularity lemma, and an influential result in its own right. Despite much effort the Brown–Erdős-Sós conjecture is wide open even for $k = 4$. For further reading please see below.

References

[1] R. Nenadov, B. Sudakov and M. Tyomkyn. Proof of the Brown-Erdős-Sós conjecture in groups. *Mathematical Proceedings of the Cambridge Philosophical Society*, 169(2) (2020), 323–333. arXiv:1902.07614.

[2] A. Shapira and M. Tyomkyn. A Ramsey variant of the Brown-Erdős-Sós conjecture. *Bulletin of the London Mathematical Society* 53(5) (2021), 1453–1469. arxiv:1910.13546

[3] A. Shapira and M. Tyomkyn. A new approach for the Brown-Erdős-Sós problem. *Israel Journal of Mathematics*. arxiv:2301.07758