7th problem set for Probability and Statistics — 22 April

Formula Summary

- Relationship between joint density and joint cumulative distribution function

$$F_{X,Y}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{X,Y}(s,t)dtds$$
$$f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x \partial y}$$

- Marginal density from joint density

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy$$
$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx$$

- **Independence:** $X \perp Y \iff F_{X,Y}(x,y) = F_X(x)F_Y(y) \iff f_{X,Y}(x,y) = f_X(x)f_Y(y)$
- **Law of Total Expectation:** $\mathbb{E}(g(X)|B) = \int_{-\infty}^{\infty} g(x)f_{X|B}(x)dx$
- **Convolution formula:** For continuous independent random variables X, Y, the variable $Z = X + Y$ has the density

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x)f_Y(z-x)dx.$$

Cumulative Distribution Function and Independence

1. We say that a random variable X (or its distribution) is *memoryless* if

$$P(X > s + t \mid X \geq s) = P(X > t)$$

for $s, t \geq 0$. In other words, the time we have already waited does not affect the time we will still wait. In the third tutorial, we saw that the geometric distribution is memoryless. Show that the exponential distribution is also memoryless. More is true: it is the only continuous memoryless distribution on positive numbers (and the geometric distribution is the only discrete one without memory), but you don't have to prove that.

2. Let $X_i \sim Exp(\lambda_i)$ for $i = 1, \ldots, n$ be independent random variables. Denote $M = \min(X_1, \ldots, X_n)$. Show that $M \sim Exp(\lambda_1 + \cdots + \lambda_n)$.

3. Let Y be the maximum of n uniformly distributed numbers in the interval $[0, 1]$.
(a) Find the cumulative distribution function F_Y.
(b) From there, determine the density function f_Y.
(c) Calculate $\mathbb{E}(Y)$.
(d) How about for the minimum of those numbers?
(e) * And what about for the k-th smallest number?

Joint Density and Total Probability

4. Let X, Y have a joint density $f_{X,Y}(x,y) = e^{-x-y}$ for $x, y > 0$ (and 0 otherwise).
(a) Determine the marginal densities f_X, f_Y.
(b) Also find the cumulative distribution functions F_X, F_Y, $F_{X,Y}$.

(c) Are X, Y independent?
(d) Find $P(X+Y \leq 1)$ and $P(X > Y)$.

5. For independent continuous random variables $X \sim U(0,2)$ and $Y \sim U(0,1)$, we examine $P(X < Y)$. Compute it:
(a) Directly from the graph.
(b) Using $P(X < Y) = \mathbb{E}(\mathbb{1}_{\{X<Y\}})$, applying LOTUS to the function $g(x,y) = \mathbb{1}_{\{x<y\}}$, and computing the integral. Do it for both orders of integration.

6. For a certain problem, we have two algorithms, A and B. Algorithm C consists of randomly choosing which of the algorithms A or B to use – A has a probability of p, and B has a probability of $1-p$, and then using this algorithm. We interpret the running time of A, B, C as random variables, denoted X, Y, Z.
(a) Determine F_Z using F_X, F_Y.
(b) If X, Y are continuous, determine f_Z using f_X, f_Y.

Convolution

7. Let $X, Y, Z \sim U(0,1)$ be independent random variables.
(a) What is the distribution of $X+Y$? Determine the density (in two ways) – using the convolution formula and by "looking at the picture".
(b) What is the distribution of $X+Y+Z$? For simplicity, determine the density function only on the interval $[0,1]$.

8. Let $X, Y, Z \sim Exp(\lambda)$ be independent random variables.
(a) What is the distribution of $X+Y$?
(b) What is the distribution of $X+Y+Z$?

For Practice

9. Choose a point uniformly randomly from the semicircle with radius 1, centered at the origin, in the upper half-plane. (Uniformly means that the probability of each subset is proportional to its area.) Let X, Y be the coordinates of the chosen point.
(a) Find the joint density $f_{X,Y}$.
(b) Find the marginal density f_Y and compute $\mathbb{E}(Y)$ using it.
(c) For verification, calculate $\mathbb{E}(Y)$ directly (using the Law of Total Expectation).

10. We break a meter stick at a point chosen uniformly at random and keep the left piece. Its length is denoted by Y. Again, we select a point uniformly at random in it, where we break it, and denote the length of the left piece by X.
(a) Find the joint density $f_{X,Y}$. The so-called conditional density $f_{X|Y} = f_{X,Y}/f_Y$ might help you.
(b) Find the marginal density f_X.
(c) Calculate $\mathbb{E}(X)$ using f_X.
(d) Calculate $\mathbb{E}(X)$ using the relationship $X = Y \cdot (X/Y)$.

11. (**Buffon's Needle**) We throw a needle of length ℓ onto an infinite floor. The floor consists of boards, with edges forming parallel lines at a distance $d \geq \ell$. Determine the probability that the needle will cross the edge of any board.