

6th problem set for Probability and Statistics — 8 April

Recall that the cumulative distribution function F_X is defined by

$$F_X(x) = P(X \leq x).$$

If X is continuous, then

$$F_X(x) = \int_{-\infty}^x f_X(t) dt$$

for a non-negative function f_X (the density of X). Then

$$P(X \in A) = \int_A f_X(t) dt, \quad \text{thus} \quad P(a \leq X \leq b) = \int_a^b f_X(t) dt$$

Also, $\mathbb{E}(X) = \int_{-\infty}^{\infty} x f_X(x) dx$ and in general

$$\mathbb{E}(g(X)) = \int_{-\infty}^{\infty} g(t) f_X(t) dt.$$

Just as for discrete random variables, here also holds that $\text{var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2$.

Before solving the problems, you might want to recall how to compute definite integrals using primitive functions.

PDF & CDF

- For a random variable X with the distribution function F_X , express
 - $P(X \in (0, 1])$
 - $P(X > 0)$
 - * $P(X < 0)$
 - * $P(X \in [0, 1])$
- For a random variable X , express the above probabilities in terms of the density function f_X .
- Let X be a random variable satisfying $P(X = x) = 0$ for every x . (Actually, there is nothing strange about this, and in fact it happens for every continuous random variable.)

Express the distribution function of the following random variables using F_X

 - $-X$.
 - $X^+ = \max(0, X)$,
 - $|X|$.
- Let X be a random variable with density $f_X(t) = 1/t^2$ for $t \geq 1$ and $f_X(t) = 0$ otherwise.
 - Verify that this is a probability density function.
 - Determine $\mathbb{E}(X)$.
 - Compute the cumulative distribution function F_X .
 - Determine $P(2 \leq X \leq 3)$.
 - Let $Y = 1/X$. What is the cumulative distribution function of the random variable Y ?
 - Determine the probability density function of the random variable Y .
- We say that X has an exponential distribution, $X \sim \text{Exp}(\lambda)$, if

$$f_X(x) = \lambda e^{-\lambda x} \quad \text{for } x \geq 0, \text{ otherwise } 0.$$

Find $F_X(x)$. * Show that $\mathbb{E}(X) = 1/\lambda$.

Continuous distributions

6. Let's assume that at a post office counter, the time for serving one customer follows an exponential distribution with an average of 4 minutes.

- What is the parameter λ ?
- Describe the distribution function.
- What is the probability that we will wait for more than 4 minutes?
- What is the probability that we will wait between 3 and 5 minutes?

7. Mr. Chen visited Prague and at a uniformly random time (0:00-24:00), he appears in the Old Town Square. Every hour from 9:00 to 23:00, 12 apostle figures appear on the astronomical clock.

- What is the probability that Mr. Chen will see the apostles without waiting for more than 15 minutes?
- What if Mr. Chen arrives at the Old Town Square at a uniformly random time after noon, i.e., 12:00–24:00?

8. We will model the amount of snow that will lie on the ground in a Krkonoše ski resort, on New Year's eve. We will use normal distribution with a mean of 40 (centimetres) and a standard deviation of 10.

- What is the probability that the model will give us a negative value for the snow cover?
- What is the probability that the snow cover will be between 50 and 70 cm?

9. We break a one-meter stick into two pieces, at a uniformly random point. Let X be the length of the longer piece.

- What is the distribution of X ?
- Determine $\mathbb{E}(X)$.

More practice problems

10. The average lifespan of a hard disk is 4 years. Let's assume that this time is described by a random variable with an exponential distribution. (This is not a realistic assumption, see e.g., <https://www.backblaze.com/blog/how-long-do-disk-drives-last/>.)

- What is the probability that the disk will fail within the first three years?
- What is the probability that it will last at least 10 years?
- After what time will the disk have failed with probability 10%?

11. Plutonium-238 has a half-life of 87.7 years. We will model its decay using an exponential distribution: for each atom, we consider the time until decay as an independent random variable with the distribution $Exp(\lambda)$.

- What is λ ?
- What is the average lifespan of a plutonium-238 atom?
- After how much time will 90% of the atoms decay?
- What percentage of atoms will decay after 50 years? (Some cardiac pacemakers use plutonium-238 as an energy source. https://en.wikipedia.org/wiki/Plutonium-238#Nuclear_powered_pacemakers)

12. The time until we see a meteor is exponentially distributed with a mean of 1 minute.

- What is the probability that we will have to wait more than 5 minutes?
- What is the probability that we will see it within at most one minute?
- * What is the distribution of the time when we see the second meteor? The third, ... (We assume that individual meteors are independent.)