6th problem set for Probability and Statistics — April 1/2

Recall that the cumulative distribution function F_X is defined by

$$F_X(x) = P(X \le x).$$

If X is continuous, then

$$F_X(x) = \int_{-\infty}^{x} f_X(t)dt$$

for a non-negative function f_X (the density of X). Then

$$P(X \in A) = \int_A f_X(t)dt, \qquad \text{thus} \qquad P(a \le X \le b) = \int_a^b f_X(t)dt$$

Also, $\mathbb{E}(X) = \int_{-\infty}^{\infty} x\ f_X(x)dx$ and in general

$$\mathbb{E}(g(X)) = \int_{-\infty}^{\infty} g(t)\ f_X(t)dt.$$

Just as for discrete random variables, here also holds that $\mathrm{var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2$.

Before solving the problems, you might want to recall how to compute definite integrals using primitive functions.

PDF & CDF

1. For a random variable X with the distribution function F_X, express
(a) $P(X \in (0,1])$ (b) $P(X > 0)$ (c) * $P(X < 0)$ (d) * $P(X \in [0,1])$

2. For a random variable X, express the above probabilities in terms of the density function f_X.

3. Let X be a random variable satisfying $P(X = x) = 0$ for every x. (Actually, there is nothing strange about this, and in fact it happens for every continuous random variable.)
Express the distribution function of the following random variables using F_X
(a) $-X$. (b) $X^+ = \max(0, X)$, (c) $|X|$.

4. Let X be a random variable with density $f_X(t) = 1/t^2$ for $t \ge 1$ and $f_X(t) = 0$ otherwise.
(a) Verify that this is a probability density function.
(b) Determine $\mathbb{E}(X)$.
(c) Compute the cumulative distribution function F_X.
(d) Determine $P(2 \le X \le 3)$.
(e) Let $Y = 1/X$. What is the cumulative distribution function of the random variable Y?
(f) Determine the probability density function of the random variable Y.

5. We say that X has an exponential distribution, $X \sim Exp(\lambda)$, if

$$f_X(x) = \lambda e^{-\lambda x} \quad \text{for } x \ge 0, \text{ otherwise } 0.$$

Find $F_X(x)$. * Show that $\mathbb{E}(X) = 1/\lambda$.

Continuous distributions

6. Let's assume that at a post office counter, the time for serving one customer follows an exponential distribution with an average of 4 minutes.

(a) What is the parameter λ?
(b) Describe the distribution function.
(c) What is the probability that we will wait for more than 4 minutes?
(d) What is the probability that we will wait between 3 and 5 minutes?

7. Mr. Chen visited Prague and at a uniformly random time (0:00-24:00), he appears in the Old Town Square. Every hour from 9:00 to 23:00, 12 apostle figures appear on the astronomical clock.

(a) What is the probability that Mr. Chen will see the apostles without waiting for more than 15 minutes?

(b) What if Mr. Chen arrives at the Old Town Square at a uniformly random time after noon, i.e., 12:00–24:00?

8. We will model the amount of snow that will lie on the ground in a Krkonoše ski resort, on New Year's eve. We will use normal distribution with a mean of 40 (centimetres) and a standard deviation of 10.

(a) What is the probability that the model will give us a negative value for the snow cover?
(b) What is the probability that the snow cover will be between 50 and 70 cm?

9. We break a one-meter stick into two pieces, at a uniformly random point. Let X be the length of the longer piece.

(a) What is the distribution of X?
(b) Determine $\mathbb{E}(X)$.

More practice problems

10. The average lifespan of a hard disk is 4 years. Let's assume that this time is described by a random variable with an exponential distribution. (This is not a realistic assumption, see e.g., `https://www.backblaze.com/blog/how-long-do-disk-drives-last/`.)

(a) What is the probability that the disk will fail within the first three years?
(b) What is the probability that it will last at least 10 years?
(c) After what time will the disk have failed with probability 10%?

11. Plutonium-238 has a half-life of 87.7 years. We will model its decay using an exponential distribution: for each atom, we consider the time until decay as an independent random variable with the distribution $Exp(\lambda)$.

(a) What is λ?
(b) What is the average lifespan of a plutonium-238 atom?
(c) After how much time will 90% of the atoms decay?
(d) What percentage of atoms will decay after 50 years? (Some cardiac pacemakers use plutonium-238 as an energy source. `https://en.wikipedia.org/wiki/Plutonium-238#Nuclear_powered_pacemakers`)

12. The time until we see a meteor is exponentially distributed with a mean of 1 minute.

(a) What is the probability that we will have to wait more than 5 minutes?
(b) What is the probability that we will see it within at most one minute?
(c) * What is the distribution of the time when we see the second meteor? The third, ... (We assume that individual meteors are independent.)