1st homework set for Mathematical analysis
Due: March 20, 2024 at 10:40 AM.

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1. *[4 points]* In each of the two cases, decide if the given set A has an upper bound, a supremum, and a maximum in the given set M. Justify your answers well: if you claim that some number is rational, write it as a fraction, if you use some theorem or observation from the lecture, state it, if you claim that some number is a supremum, prove that there is no better upper bound, etc.
   (a) $A = \mathbb{Q} \cap (1, 2)$, $M = \mathbb{Q} \setminus \mathbb{Z}$.
   (b) $A = \{0.1, 0.11, 0.111, 0.1111, 0.11111, \dots\}$, $M = \mathbb{Q}$ (note that all numbers in A have a finite decimal expansion).
2. *[6 points]* Consider the following two statements about a sequence (a_n) of real numbers. For each of them, decide whether it implies the other. For each of the possible implications, either prove it, or find a counterexample.
   (a) The sequence (a_n) is bounded.
   (b) There is a number $K \in \mathbb{R}$ such that for every $m \in \mathbb{N}$ and for every $n \in \mathbb{N}$ with $n \geq m$ we have $|a_n - a_m| < K$.
3. *[4 points]* Compute the limit or prove that it does not exist.

   (a) $\displaystyle\lim_{n\to\infty} \frac{(2n+1)^{10}(2^n+5^n)}{(3^n+5^n)(3n-1)^5(2n-1)^5}$ (b) $\displaystyle\lim_{n\to\infty} \frac{\lfloor\sqrt{2n-2}\rfloor}{2\sqrt{n}+\cos n}$,

   where $\lfloor x \rfloor$ stands for the largest integer less or equal x.
4. *[6 points]* Determine whether the following series converge. Justify your answer well.

   (a) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}}\right)^n$ (b) $\displaystyle\sum_{n=1}^{\infty} (10^{10})^{\frac{n}{10^{10}}}$