Home assignment 5

Combinatorics and Graphs 1

Submission deadline: 20 December, 12:20

Give rigorous proofs to your claims. Facts from the lecture can be used without a proof.

1. Let $d_1, \ldots, d_n$ be non-negative integers with $\sum_{i=1}^{n} d_i = 2n - 2$. Determine the number of trees on $[n]$ in which for every $i \in [n]$ the degree of i equals d_i.

2. (a) Let P_4 denote the path on four vertices and three edges. Prove that for every graph G on at least five vertices, at least one of G and $\overline{G}$ contains P_4 as a subgraph. Construct a 4-vertex graph H such that neither H nor $\overline{H}$ contains P_4 as a subgraph.

(b) Let K_4^- denote the (unique up to isomorphism) graph on 4 vertices and 5 edges. Prove that for every 10-vertex graph G, at least one of G and $\overline{G}$ contains K_4^- as a subgraph. What about 9-vertex graphs?

3. Prove for every k there exists N such that in any (vertex-)colouring $\phi : [N] \to k$ there exist $x, y, z \in [N]$ that $\phi(x) = \phi(y) = \phi(z)$ and $x + y = z$. **Hint:** define a k-colouring ψ of $\binom{[N]}{2}$ via $\psi(\{x, y\}) = \phi(|x - y|)$. Now apply Ramsey's theorem.

4. A *sunflower* is a set system $(X, \mathcal{F})$ such that there is a (possibly empty) subset $Y \subseteq X$ with $A_1 \cap A_2 = Y$ for all distinct $A_1, A_2 \in \mathcal{F}$.

Let $r \in \mathbb{N}$ and let $(\mathbb{N}, \mathcal{A})$ be a set system such that $\mathcal{A}$ is infinite and all members of $\mathcal{A}$ have cardinality r.

(a) Prove that there exists an infinite $\mathcal{B} \subseteq \mathcal{A}$ and $0 \le k \le r$ such that for any distinct $B_1, B_2 \in \mathcal{B}$ we have $|B_1 \cap B_2| = k$.

(b) Use this to prove that there exists an infinite sunflower $\mathcal{S} \subseteq \mathcal{A}$.