Home assignment 3

Combinatorics and Graphs 1

Submission deadline: 22 November, 12:20

Give rigorous proofs to your claims. Facts from the lecture can be used without a proof.

1. Seven types of chemical (call them $C_1, \dots, C_7$) are to be shipped in five trucks (call them $T_1, \dots, T_5$). There are three containers storing each type of chemical, and the capacities of the trucks $T_1, \dots, T_5$ are $6, 5, 4, 3, 3$, respectively. For security reasons, no truck can carry more than one container of the same chemical. Determine whether it is possible to ship all 21 containers in the five trucks by translating the task at hand into a network flow problem and running the Ford-Fulkerson algorithm.

2. A *circulation* in a directed graph $\vec{G}$ is a flow without a source and a sink. Given a lower capacity $\ell(x, y)$ and an upper capacity $c(x, y)$ for each edge $\vec{xy}$ with $0 \leq \ell(x, y) \leq c(x, y)$, we call a circulation g *feasible* if

$$\ell(x, y) \leq g(x, y) \leq c(x, y)$$

for every edge $\vec{xy}$. Prove that there is a feasible circulation if and only if

$$\ell(A, B) \leq c(B, A)$$

for every partition of V into sets A and $B = V \setminus A$.

Hint: One direction should be straightforward. For the other one, add a source s, a sink t, and send for every vertex of G an edge to t and an edge from s. Define on the resulting digraph G^* a capacity function c^* via $c^*(x, y) = c(x, y) - \ell(x, y)$, $c^*(s, x) = \ell(V, x)$ and $c^*(x, t) = \ell(x, V)$. This should set up a 1-1 correspondence between the feasible circulations in G and flows in G^* with value $\ell(V, V)$. Then apply Max-flow min-cut.

3. Let G be a bipartite graph that has at least one edge. Prove that G has a matching of size at least $\lceil |E(G)|/\Delta(G) \rceil$, where $\Delta(G)$ denotes the maximal degree in G.

4. Let G be a bipartite graph with bipartition (A, B).

(a) Show that if G has an A-saturating matching then there exists $a \in A$ such that every edge incident with a extends to an A-saturating matching.

(b) Suppose G is connected. Show that every edge of G extends to an A-saturating matching if and only if for every $K \subseteq A$, $K \neq \emptyset, A$, we have $|K| < |N_G(K)|$.