Home assignment 1

Combinatorics and Graphs 1

Submission deadline: 25 October, 12:20

1. Use generating functions to find a closed formula for the following sequences.

(a) $a_0 = 2, a_1 = 3$; $a_{n+2} = 3a_n - 2a_{n+1}$ for all $n \geq 0$.

(b) $b_0 = 1$; $b_{n+1} = 3b_n + 2^{n+1}$ for all $n \geq 0$.

(c) $c_0 = 1$; $c_{n+1} = c_n + n + 1$ for all $n \geq 0$.

(d) $d_0 = 4$; $d_1 = 3$; $d_n = d_{n-1} + 2d_{n-2} + 3 \cdot 2^n$ for all $n \geq 2$.

2. Find the generating functions of the sequences below. Your answer should not involve any infinite sums.

(a) $1, 0, 1, 0, 1, 0, \ldots$

(b) $0, 0, 0, 3, -3, 3, -3, \ldots$

(c) $1, 2, 1, 4, 1, 8, \ldots$

(d) $-1, 1, 0, -1, 1, 0, -1, 1, 0, \ldots$

(e) $2, 1, 4, 3, 6, 5, \ldots$

(f) $1, 0, 1, 1, 0, 2, 1, 0, 4, 1, 0, 8, 1, 0, 16, \ldots$

3. Let S be the set of all strings over the alphabet $\{a, b, c, d\}$ such that a letter a never appears next to a letter b (that is, substrings ab and ba are forbidden). Using generating functions find a closed formula for the number of n-letter strings in S.

Hint: Let u_n be the number of n-letter strings in S starting with a or b, and let v_n be the number of n-letter strings in S starting with c or d. Find recursive formulas relating u_n's and v_n's, compute the corresponding generating functions, and then derive the closed formulas. Your answer should somehow involve $\sqrt{17}$.

4. Using generating functions determine the number of ways one can collect n fruits: apples, bananas, cherries and dates under the following restrictions (combined).

- The number of apples must be even.
- The number of bananas must be a multiple of 5.
- There can be at most 4 cherries.
- There can be at most one date.