Class worksheet 11: Combinatorics and Graphs 1

December 20, 2023

Name: ____

This is just to practice, no points are awarded.

1. Revise the $(7, 4, 3)_2$ code based on the Fano plane, discussed in the lecture. Using properties of the Fano plane verify that this indeed is a code of distance 3.

2. Let H be a Hadamard matrix of order n (i.e., an $n \times n$ matrix with entries ± 1, such that the columns are orthogonal)

   (a) Prove that $\begin{bmatrix} H & H \\ H & -H \end{bmatrix}$ is also a Hadamard matrix.

   (b) Find the inverse of H. Deduce that the rows of H are orthogonal. Conclude that H^T is also a Hadamard matrix.

   (c) Prove that if $n > 2$, then n is a multiple of 4. **Hint:** First prove that n is even. Then show that without loss of generality one may assume the first row to be all 1's and the second row to be $1, \ldots, 1, -1, \ldots, -1$.

3. Let $\ell \geq 2$ be an integer, and set $n = 2^\ell - 1$, $k = 2^\ell - \ell - 1$, and $d = 3$. Suppose C is a $(n, k, d)_2$-code over $\Sigma = \{0, 1\}$. Prove that for all $\mathbf{x} \in \Sigma^n$ there exists a unique codeword $\mathbf{c} \in C$ such that $d(\mathbf{x}, \mathbf{c}) \leq 1$. **Hint:** What is the number of words in Σ^n at Hamming distance at most 1 from a codeword in C?

4. Let $\Sigma = \{0, 1, 2\}$.

   (a) Show that if a code $C \subseteq \Sigma^4$ **corrects** one error, then $|C| \leq 9$. More precisely, assume that a code $C \subseteq \Sigma^4$ has a property that for all $\mathbf{w} \in \Sigma^4$ there is at most one codeword $\mathbf{x} \in C$ such that $d(\mathbf{x}, \mathbf{w}) \leq 1$. Prove that $|C| \leq 9$.

   (b) Exhibit a code $C \subseteq \Sigma^4$ that has at least 20 codewords and **recognizes** one error. More precisely, exhibit a code $C \subseteq \Sigma^4$ with $|C| = 20$ and satisfying $d(\mathbf{x}, \mathbf{y}) > 1$ for any distinct $\mathbf{x}, \mathbf{y} \in C$.