

Class worksheet 8: Combinatorics and Graphs 1

November 29, 2023

Name: _____

This is just to practice, no points are awarded.

1. On a set X of size 5 (e.g. $X = [5]$) write down two antichains of size 10. Write down an antichain of size 8 whose members do not all have the same size.
2. Let $\mathcal{F} \subseteq 2^{[n]}$ be an antichain, and for each $k = 0, 1, \dots, n$ let $\mathcal{F}_k := \{A \in \mathcal{F} : |A| = k\}$. Prove that

$$\sum_{k=0}^n \frac{|\mathcal{F}_k|}{\binom{n}{k}} \leq 1.$$

3. Suppose that $\mathcal{F} \subseteq 2^{[n]}$ is a set systems not containing a chain of length three (that is, no distinct $A, B, C \in \mathcal{F}$ satisfy $A \subseteq B \subseteq C$). Prove that
 - (a) If n is odd then $|\mathcal{F}| \leq 2\binom{n}{\lfloor n/2 \rfloor}$. Give an example attaining this bound.
 - (b) If n is even then $|\mathcal{F}| \leq \binom{n}{n/2} + \binom{n}{n/2-1}$. Give two examples attaining this bound.
4. How many Sperner families (X, \mathcal{F}) with $|X| = n$ and $|\mathcal{F}| = \binom{n}{\lfloor n/2 \rfloor}$ are there?
5. (*) Prove that for $n \geq 5$ every graph with n vertices and $\lfloor n^2/4 \rfloor + 2$ edges contains a ‘bowtie’ (i.e., two triangles sharing exactly one vertex) as a subgraph.