Class worksheet 6: Combinatorics and Graphs 1

November 15, 2023

Name: ____

This is just to practice, no points are awarded.

1. Let $\mathcal{S} = \{S_1, \ldots, S_n\}$ be a set system on a ground set X. A *transversal* is a set of distinct $x_1, \ldots, x_n \in X$ such that $x_i \in S_i$ for all $i \in [n]$. Find a necessary and sufficient condition for existence of a transversal. **Hint:** Define appropriately an auxiliary bipartite graph, in order to apply Hall's theorem in it.

2. Consider the network (G, s, t, c) defined in yesterday's lecture, without assuming Hall's condition in G. What statement do we obtain, applying Max flow - Min cut?

3. Let G be a bipartite graph on a bipartition (X, Y) with $|X| = |Y|$. Without invoking Hall's theorem, show that if Hall's condition is satisfied in X, it is also satisfied in Y.

4. (*) Use Tutte's theorem to prove the 'hard' direction of Hall's theorem. That is, let $G = (V, E)$ be a bipartite graph with bipartition (X, Y), and assume that all sets $A \subseteq X$ satisfy $|A| \leq |N(A)|$. Using Tutte's theorem prove that G has an X-saturating matching.

   Hint: First reduce the problem to the case $|X| = |Y|$ (if not, add some vertices to G in a convenient way). Then show that for each $S \subseteq V$, $S \neq V$, the number of components C of $G \setminus S$ such that $|X \cap C| \neq |Y \cap C|$ is at most $|S|$. Now what?

5. (*) Let G be a countably infinite bipartite graph with bipartition (X, Y) such that $|N(A)| \geq |A|$ for every $A \subseteq X$. Give an example to show that G need not contain an X-saturating matching. Show however that if all vertices in X have finite degree, then G does contain an X-saturating matching.