Class worksheet 5: Combinatorics and Graphs 1

November 9, 2023

Name: ______________________________

This is just to practice, no points are awarded.

1. Attached are two past exam problems. Find a max flow and a min cut using augmenting paths or otherwise.

2. Let (G, s, t, c) be a network that has more than one maximum flow. Prove that (G, s, t, c) has infinitely many maximum flows.

3. Let $\vec{G} = (V, \vec{E})$ be a directed graph and let c be an 'extended-real' capacity function on $\vec{E}$, that is, $c(x, y)$ is a non-negative real or $+\infty$. Let s and t be two vertices. Prove that either there is a (extended-real) flow from s to t with infinite value or there is a flow with maximal finite value.

4. (*) A *circulation* in a directed graph $\vec{G}$ is a flow without a source and a sink. Given a lower capacity $\ell(x, y)$ and an upper capacity $c(x, y)$ for each edge $\vec{xy}$ with $0 \leq \ell(x, y) \leq c(x, y)$, we call a circulation g *feasible* if

$$\ell(x, y) \leq g(x, y) \leq c(x, y)$$

for every edge $\vec{xy}$. Prove that there is a feasible circulation if and only if

$$\ell(A, B) \leq c(B, A)$$

for every partition of V into sets A and $B = V \setminus A$.

Hint: One direction should be obvious. For the other one, add a source s, a sink t, and send for every vertex of G an edge to t and an edge from s. Define on the new graph G^* a capacity function c^* via $c^*(x, y) = c(x, y) - \ell(x, y)$, $c^*(s, x) = \ell(V, x)$ and $c^*(x, t) = \ell(x, V)$. Show that there is a feasible circulation in G if and only if there is a flow in G^* with value $\ell(V, V)$. Then apply Max-flow min-cut.