# Class worksheet 4: Combinatorics and Graphs 1

1 November, 2023

Name: ______________________________

This is just to practice, no points are awarded. $[n]$ is shorthand for $\{1, \dots, n\}$.

1. Prove all projective planes of order 2 are isomorphic (to this end, define isomorphism of finite projective planes appropriately).

2. Let $A$ and $B$ be a orthogonal $n \times n$ Latin squares and $\pi$ and $\sigma$ be permutations on $[n]$. Verify that $\pi(A)$ and $\sigma(B)$ are Latin squares. Show that they are orthogonal.

3. Let $A_1, \dots, A_{n-1}$ be pairwise orthogonal $n \times n$ Latin squares. Prove that for every $i, j, k, \ell \in [n]$ with $i \neq k$ and $j \neq \ell$ there exists $t \in [n-1]$ such that $A_t(i,j) = A_t(k,\ell)$. Show that such $t$ is unique.

4. Let $\mathbb{F}$ be the finite field of order $n$. How many distinct lines (through the origin) are there in $\mathbb{F}^3$?

5. (*) How many different set systems $\mathcal{F} \subset 2^{[7]}$ forming a Fano plane are there? How many automorphisms does the Fano plane have? [1]

6. (*) Define a *liberated square* of order $n$ to be an $n \times n$-table, filled with entries from $[n]$, but with no restrictions to rows or columns. Orthogonality of liberated squares is defined in the same way as for Latin squares. For a given $t$ consider the following two conditions

   (i) There exist $t$ pairwise orthogonal Latin squares of order $n$

   (ii) There exist $t+2$ pairwise orthogonal liberated squares of order $n$.

   Prove that

   (a) (i) implies (ii).

   (b) (ii) implies (i).

[1] An automorphism of a finite projective plane $(X, \mathcal{F})$ is a bijection $\sigma : X \to X$ such that for each $L \in \mathcal{F}$, $\sigma[L] := \{\sigma(x) : x \in L\}$ satisfies $\sigma[L] \in \mathcal{F}$.