Class worksheet 1: Combinatorics and Graphs 1

October 4, 2023

Name: ______________________________

This is just to practice, no points are awarded. $\mathbb{N} = \{1, 2, \dots\}$, log with unspecified base is the natural logarithm.

1. For two functions $f, g : \mathbb{N} \to \mathbb{R}$ determine (as $n \to \infty$) if $f = O(g)$, $g = O(f)$ or both
   (a) $f(n) = n^2$, $g(n) = n^3$
   (b) $f(n) = c^n$, $g(n) = n^k$, where $c > 1$ and $k \geq 1$ are constants.
   (c) $f(n) = \log_2 n$, $g(n) = \log_8 n$
   (d) $f(n) = n^3 \log_2 n$, $g(n) = 3n \log_8 n + 1$
   (e) $f(n) = (\log n)^{\log 3}$, $g(n) = (\log 3)^{\log n}$
2. Use Stirling's formula to estimate
   (a) $2 \cdot 4 \cdots 2n$
   (b) $1 \cdot 3 \cdots (2n - 1)$
   (c) $\lim_{n\to\infty} \frac{1 \cdot 3 \ldots (2n-1)}{2 \cdot 4 \cdots 2n}$
3. Using the weak form of Stirling's formula (ignoring the $\sqrt{2\pi n}$ term) estimate for large n and fixed $1 > \alpha > 0$ the value of $\binom{n}{\alpha n}$.
4. Prove that for $k = o(\sqrt{n})$ we have $\binom{n}{k} = (1 + o(1))\frac{n^k}{k!}$.
5. (*) Show that the number of subsets of $\{1, \dots, n\}$ of even cardinality is 2^{n-1}. If n is divisible by 8, what is the number of subsets of cardinality divisible by 4? (*Hint:* consider $(1 + i)^n$, where $i = \sqrt{-1}$ is the imaginary unit.)
6. (*) Let $A_1, \dots, A_n$ be finite sets. Recall the principle of inclusion-exclusion used to express $|\bigcup A_i|$. Show that if you take into account only the first $m < n$ sums in the formula, you will get an overestimate when m is odd and an underestimate when m is even.