NDMI113 – Extremal Combinatorics

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Lecture 9 - Weak saturation asymptotics.

Recall that for two r-uniform graphs G and H we say that G is *weakly H-saturated* if the missing edges on $V(G)$ can be added one by one, creating a new copy of H at every step. The weak saturation number of H, denoted $wsat(n, H)$, is the smallest number of edges in a weakly H-saturated hypergraph G on n vertices.

We have previously studied the case when H is a clique. But what can be said about weak saturation numbers in general, in particular, what is their order of magnitude? Let us first look at $r = 2$, i.e. when H is a graph, and let us assume that H has no isolated vertices. It is easy to see that $wsat(n, H) = \Theta_H(n)$ unless H has a leaf.

Lemma 1 *Let H be a graph of minimum degree $\delta \geq 2$. Then*

$$\frac{\delta - 1}{2} \cdot n \leq wsat(n, H) \leq (\delta - 1)n + O_H(1).$$

Proof For the upper bound observe that adding a new vertex of degree $\delta - 1$ to K_{n-1} creates a weakly H-saturated graph on n vertices. Therefore, for a general n, we can start with a clique K_H and successively add vertices of degree $\delta - 1$ until we have n vertices. This will give a weakly H-saturated n-vertex graph G of size $(\delta - 1)n + O_H(1)$.

For the lower bound, observe that a weakly H-saturated graph G must satisfy $\delta(G) \geq \delta - 1$ (then the bound follows by the handshaking lemma). Indeed, if G has a vertex v of degree at most $\delta - 2$, no missing edge incident with v can be ever added creating a copy of H. □

The upper bound is tight, for example, for cliques. The lower bound can, with some extra effort, be improved to $(\frac{\delta}{2} - \frac{1}{\delta+1})n$, which again is tight. In this light, it is natural to ask if the weak saturation number is linear in the stronger sense that $\lim_{n\to\infty} wsat(n, H)/n$ exists. This was proved true by Alon ('85) using the following important tool.

Theorem 1 (Fekete's subadditivity lemma) *Let $(a_n)_{n=1}^{\infty}$ be a sequence of non-negative reals satisfying $a_{m+n} \leq a_m + a_n$ for all m and n. Then*

$$\lim_{n\to\infty} \frac{a_n}{n} = \inf_n \frac{a_n}{n},$$

in particular, the limit $\lim_{n\to\infty} \frac{a_n}{n}$ always exists.[1]

[1] The assumption of non-negativity is not crucial. In general the limit always exists if we allow $-\infty$ as its value.

Proof Let $c = \inf_n \frac{a_n}{n}$, fix some $\epsilon > 0$ and let m be such that $\frac{a_m}{m} \leq c + \epsilon$; it exists since c is the infimum. Note that, by subadditivity, when n is an integer multiple of m, $n = km$, we have

$$\frac{a_n}{n} = \frac{a_{km}}{km} \leq \frac{ka_m}{km} = \frac{a_m}{m} \leq c + \epsilon.$$

We now claim that for *all* sufficiently large n we have $\frac{a_n}{n} \leq c + 2\epsilon$.

To this end, suppose

$$k > \epsilon^{-1} \cdot \max_{r \in [m-1]} a_r,$$

and let $n = km + r$, where $r \in [m-1]$ (we know already that the statement holds for $r = 0$). We obtain

$$\begin{aligned} a_n = a_{km+r} &\leq ka_m + a_r \leq km(c+\epsilon) + a_r \leq km(c + 2\epsilon) \\ &\leq (km + r)(c + 2\epsilon) = n(c + 2\epsilon). \end{aligned}$$

□

Note that the first index of the sequence need not be 1, the proof is similar for any starting index.

Theorem 2 (Alon '85) *Let H be a graph with $\delta(H) > 1$. There exists a constant $c_H > 0$ such that*

$$wsat(n, H) = (c_H + o(1)) \cdot n.$$

Note that when $\delta(H) = 1$ we have $wsat(n, H) = O_H(1)$, by Lemma 1.

Alon's theorem is a consequence of the Fekete lemma and the following subadditive property of the graph weak saturation function.

Lemma 2 *For any $m, n \geq |H| = |V(H)|$ we have*

$$wsat(n + m, H) \leq wsat(m, H) + wsat(n, H) + |H|^2.$$

With this lemma in hand we proceed as follows. Let $f(n) = wsat(n, H) + |H|^2$. Then, by Lemma 2,

$$f(n + m) = wsat(n + m, H) + |H|^2 \leq wsat(m, H) + wsat(n, H) + 2|H|^2 = f(m) + f(n).$$

So, the function f is subadditive, which by Theorem 1 implies the existence of $\lim_{n\to\infty} \frac{f(n)}{n}$. This in turn implies the existence of $c_H = \lim_{n\to\infty} \frac{wsat(n,H)}{n}$. Lemma 1 then gives $c_H > 0$.

Proof [of Lemma 2] Let F_1 and F_2 be weakly H-saturated graphs with m and n vertices, and $wsat(m, H)$ and $wsat(n, H)$ edges, respectively. Take their disjoint union, and let $V_1 \subset V(F_1)$ and $V_2 \subseteq V(F_2)$ be designated sets of $|H|$ vertices. Add all edges between V_1 and V_2, and let G be the resulting graph. It has $m + n$ vertices and $wsat(n + m, H) + |H|^2$ edges.

To see that G is weakly H-saturated observe that we can first apply weak saturation in F_1 and F_2 separately, resulting in two cliques on $V(F_1)$ and $V(F_2)$. Then we can fill the edges with one vertex in V_1 or V_2: adding each of them creates a new copy of $K_{|H|}$, and therefore also of H. Lastly, we can fill the remaining edges: each of them also completing a $K_{|H|}$. □

What about higher uniformities? When H is an r-uniform clique, we have seen that $wsat(n, H)$ is of order of magnitude n^{r-1}. This suggests a general pattern, but in order to phrase the generalizations of Lemma 1 and Theorem 2 we need to formulate the hypergraph counterpart of the "not having a leaf" graph property.

Definition 1 (Sharpness) *Let H be an r-uniform hypergraph. The* sharpness *of H is*

$$s(H) = \min\{|S| : S \subseteq V, |\{e \in E(H) : S \subseteq e\}| = 1\}.$$

In words, the sharpness is the smallest size of a vertex set contained in precisely one edge. So for graphs, the sharpness is 1 if the graph contains a leaf, and 2 otherwise. And in general we have $1 \leq s(H) \leq r$, whereby $s(H) = r$ is the 'generic' case. Tuza observed that the sharpness is the governing parameter for the coarse asymptotics of the *wsat* function.

Lemma 3 (Tuza '92) *Let H be an r-uniform hypergraph of sharpness s. Then*

$$wsat(n, H) = \Theta_H(n^{s-1}).$$

The proof is analogous to that of Lemma 1, we leave it as an exercise. Tuza then went on to conjecture that, similarly to Theorem 2, there is a limiting constant in the n^{s-1} term. This was proven in 2023.

Theorem 3 (Shapira-T. '23) *Let H be an r-uniform hypergraph of sharpness s. Then there is a constant $c_H > 0$ such that*

$$wsat(n, H) = (c_H + o(1))n^{s-1}.$$

We present a simplified proof due to Terekhov ('25), which uses the same two ideas at its core. The first idea is that, instead of attempting to come up with a "polynomial Fekete lemma" that we could use as a black box, we should adapt the *proof method* of the original Fekete lemma to suit our needs.

The second proof idea is to split not the vertex set of the target graph G like we did in the proof of Theorem 2, but the set of all $(s-1)$-tuples of vertices, appropriately. To this end, we shall apply the following well-known result, whose proof is probabilistic and outside our scope.

Theorem 4 (Rödl '85) *For any $k \geq t \geq 0$ and $\delta > 0$ there exists $N_0(k, t, \delta) \geq k$ such that for any set X of size $|X| \geq N_0(k, t, \delta)$ there exists a family $\mathcal{F}_X \subseteq \binom{X}{k}$ of size $|\mathcal{F}_X| \leq (1 + \delta)\binom{|X|}{t}/\binom{k}{t}$, such that every $T \in \binom{X}{t}$ is contained in some $A \in \mathcal{F}_X$.*

Proof [of Theorem 3] For notational simplicity let us assume that $r = s = 3$, so we know from Lemma 3 that $wsat(n, H) = \Theta(n^2)$ and are now trying to prove that $wsat(n, H) = (c_H + o(1))n^2$ for a constant $c_H > 0$.

We put $v = |V(H)|$, define

$$c = \liminf_n \frac{wsat(n, H)}{\binom{n-v}{2}},$$

and note that by Lemma 3 we have $c > 0$.

Fix $\epsilon > 0$. By the above definition of c there exists $m \geq v + 2$ such that

$$wsat(m, H) \leq (c + \epsilon)\binom{m - v}{2}.$$

Let F_0 be a weakly H-saturated hypergraph with m vertices and $wsat(m,H)$ many edges. We now claim that for all sufficiently large n there exists a weakly H-saturated graph G with $|V(G)| = n$ and

$$|E(G)| \leq (1+\epsilon)(c+\epsilon)\binom{n-v}{2}.$$

This would readily imply the existence of the limit $c = \lim_{n\to\infty} wsat(n,H)/\binom{n-v}{2}$, meaning

$$\lim_{n\to\infty} \frac{wsat(n,H)}{n^2} = \frac{c}{2} =: c_H.$$

By Theorem 4, applied with $k = m - v$, $t = 2$ and $\delta = \epsilon$, there exists[2] $n_0 \geq m - v$ such that for any set X of size at least n_0 there exists a family $\mathcal{F}_X \subseteq \binom{X}{m-v}$ such that each pair in X is a subset of some $A \in \mathcal{F}_X$, and

$$|\mathcal{F}_X| \leq (1+\epsilon)\frac{\binom{|X|}{2}}{\binom{m-v}{2}}.$$

Now let $n \geq n_0 + v$, let $Z = [v]$, $X = [n] \setminus [v]$, and let $\mathcal{F}_X \subseteq \binom{X}{m-v}$ be as stated above. For every $A \in \mathcal{F}_X$ put a copy of F_0 on $A \cup Z$, and let G be the resulting 3-uniform hypergraph on $[n]$. Observe that

$$|E(G)| \leq |\mathcal{F}_X| \cdot |E(F_0)| \leq (1+\epsilon)\frac{\binom{|X|}{2}}{\binom{m-v}{2}}(c+\epsilon)\binom{m-v}{2} = (1+\epsilon)(c+\epsilon)\binom{n-v}{2}.$$

We now aim to show that G is weakly H-saturated. To see this, first we run the saturation process inside each copy of F_0, filling all the edges on the vertex set $A \cup Z$ for each $A \in \mathcal{F}_X$. By the definition of $\mathcal{F}_X$ we now have that each $e \in \binom{[n]}{3}$ with $|e \cap X| \leq 2$ has been filled, since $e \cap X$ was a subset of some $A \in \mathcal{F}_X$, so $e \subseteq A \cup Z$. It remains to add the missing vertex triples that are subsets of X. For this, let f be any such triple, and observe that the vertices of f together with any $v-3$ vertices from Z form a set $W \in \binom{[n]}{v}$ such that each $e \in \binom{W}{3} \setminus \{f\}$ satisfies $|e \cap X| \leq 2$, and therefore has already been filled. Thus, adding f would create a new copy of K_v, and, a fortiori, a new copy of H. □

Exercise 1 *Do the proof for arbitrary r and s. In particular, the last step requires more attention.*

Many questions about the behaviour of the limiting constant remain open.

Conjecture 1 $c_H \in \mathbb{Q}$.

Conjecture 2 (Tuza '92) $wsat(n,H) = c_H(n^{s-1}) + O(n^{s-2})$.

More generally, how is c_H related to the structure of H?

[2] For $t = s - 1 = 2$ instead of Rödl's theorem one may also refer to a more classical existence of Steiner systems.