NDMI113 – Extremal Combinatorics

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Lecture 7 - Weak saturation.

Recall from Lecture 4 that for two r-uniform hypergraphs G and H we say that G is H-saturated if G does not contain H as a subgraph, but upon adding any missing edge to G the resulting hypergraph *will* contain H as a subgraph. The saturation number $\mathrm{sat}(n, H)$ is the minimal number of edges in an H-saturated hypergraph G on n vertices.

Bollobás's theorem links the saturation number of cliques to the Two families theorem. We restate them here for the reader's convenience.

Theorem 1 (Two families theorem, uniform case)
Let $\mathcal{A} = \{A_1, \dots, A_m\} \subseteq \binom{\mathbb{N}}{a}$ and $\mathcal{B} = \{B_1, \dots, B_m\} \subseteq \binom{\mathbb{N}}{b}$ be two set systems such that

- $A_i \cap B_i = \emptyset$, *for all* i, *and*
- $A_i \cap B_j \neq \emptyset$, *for all* $i \neq j$.

Then

$$m \leq \binom{a+b}{b}.$$

Theorem 2 *Let G be an r-uniform hypergraph on $[n]$, and suppose that adding any missing edge to G creates a copy of $K^{(r)}_{r+s}$. Then*

$$e(G) \geq \binom{n}{r} - \binom{n-s}{r}.$$

In particular, we have $sat(n, K^{(r)}_{r+s}) = \binom{n}{r} - \binom{n-s}{r}$.

Bollobás conjectured that the uniform Two families theorem's assertion holds even under the weaker second assumption that $A_i \cap B_j \neq \emptyset$ for all $i < j$ (note the asymmetry). This was confirmed by Lovász.

Theorem 3 (Skewed two families theorem - Lovász '77) *Let $\mathcal{A} = \{A_1, \dots, A_m\} \subseteq \binom{\mathbb{N}}{a}$ and $\mathcal{B} = \{B_1, \dots, B_m\} \subseteq \binom{\mathbb{N}}{b}$ be two set systems such that*

- $A_i \cap B_i = \emptyset$, *for all* i, *and*
- $A_i \cap B_j \neq \emptyset$, *for all* $i < j$.

Then

$$m \leq \binom{a+b}{b}.$$

In this light it is perhaps surprising that the Two families theorem does *not* directly extend to a non-uniform skewed version (**Exercise:** construct a counterexample), although it can be extended under additional assumptions (Scott–Wilmer '21).

Going back to the skewed uniform setting, what strengthening of Theorem 2 does Theorem 3 give us?

Definition 1 (weakly saturated) *For two r-uniform hypergraphs G and H we say that G is* weakly H-saturated *if there exists an ordering $e_1, \dots, e_m$ of $\binom{V(G)}{r} \setminus E(G)$ such that for each $\ell \in [m]$ the graph $G_\ell = G \cup \{e_1, \dots, e_\ell\}$ contains a copy of H not contained in $G_{\ell-1}$ (i.e. containing the edge e_ℓ). The* weak saturation number *$wsat(n, H)$ is the minimal number of edges in a weakly H-saturated hypergraph on n vertices.*

In words: in a weakly saturated graph G the missing edges can be added one by one, with every new edge creating a new copy of H. The sequence $G = G_0, G_1, \dots, G_\ell = \binom{V(G)}{r}$ is called a *saturating sequence* or *saturation process* of G (note that it is not unique).

Example 1 *For $r = 2$ and $H = K_3$ a graph G is weakly saturated if and only if G is connected. Therefore, $wsat(n, K_3) = n - 1$.*

Theorem 4 *$wsat(n, K_{r+s}^{(r)}) = \binom{n}{r} - \binom{n-s}{r}$.*

Let us see how this is implied by Theorem 3.

Proof Let $H = K_{r+s}^{(r)}$. Suppose that $G \subseteq \binom{[n]}{r}$ is weakly H-saturated with a saturating sequence $G = G_0 \subseteq \cdots \subseteq G_\ell = \binom{[n]}{r}$. Take $A_i = E(G_i) \setminus E(G_{i-1}) \in \binom{[n]}{r}$ – note that the A_i are the e_i from Definition 1. This gives rise to a set system $\mathcal{A} = \{A_1, \dots, A_m\} \subseteq \binom{[n]}{r}$. Each A_i completes an $(r+s)$-clique on a vertex set M_i. Take $B_i = [n] \setminus M_i$, this defines a set system $\mathcal{B} = \{B_1, \dots, B_m\} \subseteq \binom{[n]}{n-r-s}$. Observe now that $\mathcal{B}$ and $\mathcal{A}$ satisfy the conditions of Theorem 3. Indeed, on the one hand we have $A_i \subseteq M_i$, and so $A_i \cap B_i = \emptyset$. While on the other hand for $i < j$ we have $A_j \not\subseteq M_i$, as otherwise A_i would not be able to complete the clique on M_i — the edge A_j would still be missing, and therefore $A_i \cap B_j \neq \emptyset$. So, Theorem 3 gives

$$m \leq \binom{r + n - s - r}{r} = \binom{n-s}{r}.$$

□

In order to prove Theorem 3, we need to introduce the exterior algebra of a vector space — Lovasz's ingenious idea was to use it here.

Definition 2 (Exterior product) *Let V be an n-dimensional real vector space.* [1] *For $2 \leq k \leq n$ the k-th exterior power of V, denoted $\Lambda^k V$, is defined as the space spanned by elements of the form*

$$v_1 \wedge v_2 \wedge \cdots \wedge v_k, \quad v_i \in V,$$

satisfying the multilinear and alternating relations.

[1] Works similarly over any field of characteristic not equal 2.

Each vector in $\Lambda^k V$ can be written as a linear combination of the vectors $e_{i_1} \wedge \cdots \wedge e_{i_k}$, where $e_1, \ldots, e_n$ are some fixed basis vectors. It is not hard to verify that $\Lambda^k V$ is a vector space of dimension $\binom{n}{k}$. For $k = 0$ and $k = 1$ we set $\Lambda^k V$ to be $\mathbb{R}$ and V, respectively.

The exterior product of the spaces $V_1, \ldots, V_k$ is defined similarly. That is, the space $V_1 \wedge \cdots \wedge V_k$ is the set of all linear combinations of elements of the form $v^1 \wedge \cdots \wedge v^k$, where $v_i \in V_i$, satisfying multilinear and alternating relations, e.g.

$$(v_1 + v_1') \wedge \cdots \wedge v_k = (v_1 \wedge \cdots \wedge v_k) + (v_1' \wedge \cdots \wedge v_k),$$

and

$$v_2 \wedge v_1 \wedge v_3 \wedge \cdots \wedge v_k = -v_1 \wedge v_2 \wedge v_3 \cdots \wedge v_k.$$

In particular, $v_1 \wedge v_2 = -v_2 \wedge v_1$ and so $v \wedge v - v \wedge v = 0$. Importantly, $v_1 \wedge \cdots \wedge v_k \neq 0$ if and only if the vectors are linearly independent. Finally, define $\Lambda V = \Lambda^0 V \oplus \cdots \oplus \Lambda^n V$. This is a 2^n-dimensional vector space.

Proof [of Theorem 3] Let $X = (\bigcup_{i \in [m]} A_i) \cup (\bigcup_{i \in [m]} B_i)$ be the underlying ground set. Let $V = \mathbb{R}^{r+s}$ and take $Z = \{z_x \mid x \in X\} \subseteq V$ to be a set of $|X|$ vectors in 'general position' (any $r+s$ among them are linearly independent). For example, take $|X|$ non-zero points on the *moment curve* $(t, t^2, \ldots, t^{r+s})$, so any matrix defined by some $r+s$ of them is essentially Vandermonde, and so is non-singular. For each $A_i = \{x_1, \ldots, x_r\} \in \mathcal{A}$ and $B_i = \{x_1, \ldots, x_s\} \in \mathcal{B}$ define $v_{A_i} = z_{x_1} \wedge \cdots \wedge z_{x_r}$ and $v_{B_i} = z_{x_1} \wedge \cdots \wedge z_{x_s}$. Then $v_{A_i} \wedge v_{B_i} \neq 0$ since $A_i \cap B_i = \emptyset$ and $A_i \cup B_i$ consists of $r+s$ points in general position. On the other hand, for $i < j$ we have $v_{A_i} \wedge v_{B_j} = 0$ since $A_i \cap B_j \neq 0$ so some z_{x_m} is used twice. We claim that $\{v_{A_i} \mid A_i \in \mathcal{A}\}$ is linearly independent in $\Lambda^r V$ (so $m = |A| = |B|$ satisfies $m \leq dim(\Lambda^r V) = \binom{r+s}{r}$). Suppose to the contrary that the set is linearly dependent, i.e. suppose

$$v = \sum_{i \in I \subseteq [m]} \lambda_i v_{A_i} = 0,$$

where all λ_i are not 0 and $I \neq 0$. Then let ℓ be the maximum element of I. We have

$$0 = v \wedge v_{B_\ell} = (\sum_{i \in I \subseteq [m]} \lambda_i v_{A_i}) \wedge v_{B_\ell} = \sum_{i \in I \subseteq [m]} \lambda_i (v_{A_i} \wedge v_{B_\ell}) = \lambda_\ell (v_{A_\ell} \wedge v_{B_\ell}),$$

but $\lambda_\ell \neq 0$ and $v_{A_\ell} \wedge v_{B_\ell} \neq 0$, a contradiction. □

To this day, all known proofs of Theorem 4 use tools from algebra or geometry — no purely combinatorial proof is known. This might be explained by the fact that the extremal examples do not exhibit any stability behaviour. For instance, in Example 1, when H is the triangle, the extremal graphs are all trees. Compare this with the (strong) saturation, where the unique extremal graph is the star. Since a combinatorial prove tends to give as a by-product some structural information about the extremal cases, we are less likely to find one for weak saturation. This does not mean it is not possible — after all, it *was* for the triangle, but one would need to work harder. Even the next case $H = K_4$ is already a challenge.

Exercise 1 *Prove combinatorially that* $wsat(n, K_4) = 2n - 3$.