NDMI113 – Extremal Combinatorics

Mykhaylo Tyomkyn

Lecture 6 - Applications of Frankl-Wilson.

In the previous lecture we covered the Modular Frankl-Wilson theorem. We restate it here for the reader's convenience.

Theorem 1 (Modular Frankl-Wilson) *Let p be a prime and $S \subseteq \{0, \ldots, p-1\}$. Let $\mathcal{F} \subseteq 2^{[n]}$ be such that $|A|_{mod\ p} \notin S$ for all $A \in \mathcal{F}$, while $|A \cap B|_{mod\ p} \in S$ for all $A \neq B \in \mathcal{F}$. Then,*

$$|\mathcal{F}| \leq \sum_{i=0}^{|S|} \binom{n}{i}.$$

In this lecture we discuss three famous applications of this theorem.

The Hadwiger-Nelson problem

Let $U(\mathbb{R}^n)$ be the unit distance graph on $\mathbb{R}^n$. That is, let $U(\mathbb{R}^n)$ be the graph with vertex set $\mathbb{R}^n$ and the edge set of all $\{x, y\}$ such that $||x - y|| = 1$, where $|| \cdot ||$ stands for the usual euclidean distance. The Hadwiger-Nelson problem asks to determine $\chi(U(\mathbb{R}^n))$, the chromatic number of this graph. By compactness, this is equivalent to finding the largest chromatic number among finite subgraphs of $U(\mathbb{R}^n)$ (or ∞ if they are unbounded).

Since the regular simplex in $\mathbb{R}^n$ forms a clique in $U(\mathbb{R}^n)$, we get a lower bound $\chi(U(\mathbb{R}^n)) \geq n + 1$ [1]. By contrast, the best known upper bound is $\chi(U(\mathbb{R}^n)) < (3 + o(1))^n$ due to Larman and Rogers ('75). The following theorem of Frankl and Wilson establishes a lower bound that is exponential as well.

Theorem 2 (Frankl-Wilson '81) *There exists $c > 1$ such that $\chi(U(\mathbb{R}^n)) > c^n$.*

We shall need the following Lemma.

Lemma 1 *Let p be a prime and let $\mathcal{F} \subseteq \binom{[4p]}{2p}$. Let $|A \cap B| \neq p$ for all $A, B \in \mathcal{F}$. Then $|\mathcal{F}| \leq 4\binom{4p}{p-1}$.*

Proof For each $x \in [4p]$, define $\mathcal{F}_x := \{A \in \mathcal{F} \mid x \in A\}$. Since

$$\sum_{x \in [4p]} |\mathcal{F}_x| = \sum_{A \in \mathcal{F}} |A| = 2p|\mathcal{F}|,$$

[1] In $\mathbb{R}^2$ a 7-vertex graph known as the Moser spindle demonstrates $\chi(U(\mathbb{R}^2)) \geq 4$. A 2018 result of de Grey improves this to $\chi(U(\mathbb{R}^2)) \geq 5$. On the other hand, by tiling the plane with hexagons, it can be seen that $\chi(U(\mathbb{R}^2)) \leq 7$.

there exists x with $|\mathcal{F}_x| \geq \frac{1}{2}|\mathcal{F}|$. Then, for this x and any distinct $A, B \in \mathcal{F}_x$ we have

$$|A \cap B| \notin \{0, p, 2p\}.$$

In other words,

$$|A \cap B| \not\equiv 0 \bmod p.$$

Since for all $A \in \mathcal{F}_x$ we clearly have $|A| \equiv 0 \bmod p$, we can apply Modular Frankl-Wilson with $S = \{1, \ldots, p-1\}$ to obtain

$$|\mathcal{F}_x| \leq \sum_{i=0}^{p-1} \binom{4p}{i} \leq 2\binom{4p}{p-1},$$

since $\binom{4p}{i} \geq 2\binom{4p}{i-1}$ for all $i \leq p-1$. Hence, $|\mathcal{F}| \leq 2|\mathcal{F}_x| \leq 4\binom{4p}{p-1}$, as desired. $\square$

Proof [of Theorem 2] Let p be a large prime and $n = 4p$. Consider the indicator vectors χ_A of the sets $A \in \binom{[4p]}{2p}$ in $\mathbb{R}^n$. Observe that

$$\|\chi_A - \chi_B\| = \sqrt{|A \Delta B|} = \sqrt{|A| + |B| - 2|A \cap B|} = \sqrt{2(2p - |A \cap B|)}.$$

In particular, $\|\chi_A - \chi_B\| = \sqrt{2p}$ if and only if $|A \cap B| = p$.
Let now $V \subseteq \mathbb{R}^n$ be the set $\{v_A = \frac{1}{\sqrt{2p}}\chi_A : A \in \binom{[4p]}{2p}\}$, and G be the unit distance graph on V.
By construction, $\{v_A, v_B\} \in E(G)$ if and only if $|A \cap B| = p$. Therefore, by Lemma 1 we have

$$\alpha(G) \leq 4\binom{4p}{p-1}.$$

It follows that

$$\frac{\binom{4p}{2p}}{4\binom{4p}{p-1}} \leq \frac{|V|}{\alpha(G)} \leq \chi(G) \leq \chi(\mathbb{R}^n).$$

Standard estimates of binomial coefficients show that the left hand side grows exponentially in p, and therefore also in n, as $n = 4p$. It can be calculated to be at least c^n with $c > 1.14$. $\square$

The Borsuk conjecture

Suppose that $K \subseteq \mathbb{R}^d$ is a set of diameter 1, [2] and define $k(K)$ to be the minimum number of sets of diameter strictly less than 1 needed to partition K. Borsuk's conjecture from 1933 stated that $k(d) := \sup_{diam(K)=1} k(K) = d + 1$. That is, every set of diameter 1 can be partitioned into at most $d+1$ sets of diameter less than 1. Borsuk's conjecture was proved for $d \leq 3$ and for general d when K a smooth convex set or a centrally symmetric set. However, the general Borsuk's conjecture disproved by Kahn and Kalai in 1993.

Theorem 3 *There exists $c > 1$ such that $k(d) > c^{\sqrt{d}}$ for large enough d.*

[2]supremal euclidean distance between any two points in K

Proof Let p be a large prime and $d = \binom{4p}{2}$. Set $W = \binom{[4p]}{2} = E(K_{4p})$, so we identify W with the edge set of the complete graph on $[4p]$. Index the coordinates of $\mathbb{R}^d$ with W (i.e. we work in $\mathbb{R}^W$). Under this viewpoint, since coordinates correspond to the edges of K_{4p}, the 0/1-vectors correspond to the subgraphs of K_{4p}.

Let $\mathcal{F} = \binom{[4p]}{2p}$. For each $A \in \mathcal{F}$, define

$$E_A = \{\{i,j\} \in W \mid |A \cap \{i,j\}| = 1\}.$$

That is, E_A is the complete bipartite graph on the vertex sets A and $A^c = [4p] \setminus \{A\}$.

Let $\mathcal{G} = \{E_A \mid A \in \mathcal{F}\}$. Since $|A| = |A^c| = 2p$ and $E_A = E_{A^c}$, we have

$$|\mathcal{G}| = \frac{1}{2}\binom{4p}{2p}.$$

Let χ_A be the indicator vector of E_A: a 0/1-vector in $\mathbb{R}^d$ defined by

$$\chi_A(e) = \begin{cases} 1 & \text{if } e \in E_A \\ 0 & \text{if } e \notin E_A. \end{cases}$$

Define $K = \{\chi_A : A \in \mathcal{F}\}$, this is a finite point set in $\mathbb{R}^d$ of size

$$|K| = |\mathcal{G}| = \frac{1}{2}\binom{4p}{2p}.$$

Let $A, B \in \mathcal{F}$ with $A \neq B, B^c$. Then

$$\|\chi_A - \chi_B\| = \left(\sum_{e \in W} (\chi_A(e) - \chi_B(e))^2\right)^{1/2} = |E_A \Delta E_B|^{1/2} = (|E_A| + |E_B| - 2|E_A \cap E_B|)^{1/2}.$$

We see that $\|\chi_A - \chi_B\|$ is maximized when $|E_A \cap E_B|$ is minimized. It can be easily checked that the latter happens when $|A \cap B| = p$. Now suppose that $L \subseteq K$ satisfies $\mathrm{diam}(L) < \mathrm{diam}(K)$. Considering the subfamily $\mathcal{G}' \subseteq \mathcal{G}$ encoding L gives rise to a corresponding set system $\mathcal{F}' \subseteq \mathcal{F}$. In $\mathcal{F}'$, there is no pair A, B with $|A \cap B| = p$, so by Lemma 1, $|\mathcal{G}'| \leq 4\binom{4p}{p-1}$. Thus, in order to partition K into sets of diameter less than one 1, one needs at least

$$\frac{\frac{1}{2}\binom{4p}{2p}}{4\binom{4p}{p-1}} \geq c^p > c'^{\sqrt{d}}$$

sets in the partition, where, it turns out, one can take $c' > 1.04$.

□

Deterministic Ramsey constructions

Modular Frankl-Wilson readily gives the following 'exact' (as opposed to 'modular') counterpart.

Theorem 4 *Let $\mathcal{F} \subseteq 2^{[n]}$ and $S \subseteq [n]$ be such that $|A| \notin S$ for all $A \in \mathcal{F}$ and $|A \cap B| \in S$ for all distinct $A, B \in \mathcal{F}$. Then*

$$|\mathcal{F}| \leq \sum_{i=0}^{|S|} \binom{n}{i}.$$

Proof Choose a prime $p > n$ and apply Modular Frankl-Wilson. $\square$

Recall the well-known lower bound for the Ramsey numbers:

$$R(t,t) \geq \sqrt{2}^{t}.$$

The proof is due to Erdős and uses a basic probabilistic (or counting) argument: include every possible edge between n vertices independently, with probability $1/2$, then, for appropriate n, with a positive probability the graph will have no clique or independent set of order t.

Tantalizingly, this is essentially still the best known bound as of year 2025. Its reliance on an 'indirect' probabilistic argument raises a natural question concerning *explicit* constructions. The disjoint union of $t-1$ cliques of order $t-1$ each demonstrates that $R(t,t) > (t-1)^2$; this is only quadratic in t and there seems to be no obvious improvement using the same idea. In this light it is striking that Frankl and Wilson were able to give a super-polynomial constructive lower bound.

Theorem 5 *$R(t,t) > t^{c \log_2 t / \log_2 \log_2 t}$ where $c > 0$ is an absolute constant, via an explicit construction.*

Proof Let p be a large prime, $r = p^2 - 1$ and $m = p^3$. Define a graph G on the vertex set $V(G) = \binom{[m]}{r}$ and with $\{A, B\} \in E(G)$ if and only if $|A \cap B| \equiv -1 \bmod p$. Using the Modular Frankl-Wilson (with $S = \{0, \ldots, p-2\}$) we can upper bound the independence number of G as follows

$$\alpha(G) \leq \sum_{i=0}^{p-1} \binom{m}{i} \leq 2\binom{m}{p-1}.$$

The clique number $\omega(G)$, in turn, can be upper bounded using Theorem 4 (with $S = \{p-1, 2p-1, \ldots, (p-1)p-1\}$):

$$\omega(G) \leq \sum_{i=0}^{p-1} \binom{m}{i} \leq 2\binom{m}{p-1}.$$

In conclusion, we have $|V(G)| = \binom{p^3}{p^2-1}$ while $\alpha(G), \omega(G) \leq t := 2\binom{p^3}{p-1}$. Applying Stirling's approximation yields

$$R(t,t) > t^{c \log_2 t / \log_2 \log_2 t},$$

for a constant $c > 0$. $\square$