NDMI113 – Extremal Combinatorics

Mykhaylo Tyomkyn

Lecture 4 - Intersecting set systems.

Definition 1 (Intersecting) *A set system $\mathcal{F} \subseteq 2^{[n]}$ is intersecting if $A \cap B \neq \emptyset$ for all $A, B \in \mathcal{F}$.*

A natural question to ask is how large an intersecting set can be.

Example 1 *The 'star' set system $\{A \subseteq [n] \mid 1 \in A\}$ is intersecting and has size 2^{n-1}. It is easy to see that this is the largest possible. Indeed, let $\mathcal{F} \subseteq 2^{[n]}$ be intersecting. Partition $2^{[n]}$ into 2^{n-1} pairs $\{A, [n] \setminus A\}$. At most one set from each pair can be in $\mathcal{F}$.*

It becomes more interesting if we ask about largest intersecting k-uniform families $\mathcal{F} \subseteq \binom{[n]}{k}$. There are three different regimes:

1. $k > n/2$: The whole layer $\binom{[n]}{k}$ is intersecting.
2. $k = n/2$: This case only occurs if n is even. Take one set from each complementary pair $\{A, [n] \setminus A\}$. Observe that any set system $\mathcal{F}$ constructed in this way is intersecting and has size
$$\frac{1}{2}\binom{n}{k} = \binom{n-1}{k-1}.$$
3. $k < n/2$: This case is the most interesting. One candidate to be the extremal example is again the 'star', i.e. the family of all k-uniform sets containing a fixed element, say 1. This gives $|\mathcal{F}| = \binom{n-1}{k-1}$. Another candidate may be to take all sets containing two elements from $\{1, 2, 3\}$. However, the former set system is larger. In fact, the following theorem guarantees that the star is optimal.

Theorem 1 (Erdős-Ko-Rado) *For $k \leq n/2$, if $\mathcal{F} \subseteq \binom{[n]}{k}$ is intersecting, then $|\mathcal{F}| \leq \binom{n-1}{k-1}$.*

We shall provide two proofs.

Proof [First proof] Let $\mathcal{G} = \{\bar{A} = [n] \setminus A \mid A \in \mathcal{F}\} \subseteq \binom{[n]}{n-k}$. Since $\mathcal{F}$ is intersecting, no $A \in \mathcal{F}$ is contained in any $\bar{B} \in \mathcal{G}$. Therefore, taking the $(n-2k)$-th shadow of $\mathcal{G}$, we obtain a k-uniform set system $\mathcal{H} = \partial^{(n-2k)}\mathcal{G}$ such that $\mathcal{F} \cap \mathcal{H} = \emptyset$, and so
$$|\mathcal{F}| + |\mathcal{H}| \leq \binom{n}{k}.$$

Now suppose that $|\mathcal{F}| > \binom{n-1}{k-1} = \binom{n-1}{n-k}$. Then, by repeated applications of Kruskal-Katona (Lovász form), we get
$$|\mathcal{H}| = |\partial^{(n-2k)}\mathcal{G}| \geq \binom{n-1}{n-k-(n-2k)} = \binom{n-1}{k}.$$

So,

$$|\mathcal{F}| + |\mathcal{H}| > \binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k},$$

a contradiction. □

Proof [Second proof] Consider an arbitrary bijection $f : [n] \to \mathbb{Z}_n$. We say that a set A 'maps to an interval' under f if $f(A) := \{f(a) \mid a \in A\} = \{i, i+1, \ldots, i+k-1\}$ for some $0 \leq i \leq n-1$, where addition is taken modulo n. We now double count N, the number of pairs (f, A) such that $f : [n] \to \mathbb{Z}_n$ and $f(A)$ is an interval.

We claim that for any given f, there are at most k sets in $\mathcal{F}$ that map to an interval under f. Fix $A \in \mathcal{A}$ and suppose that $f(A) = \{i, i+1, \ldots, i+k-1\}$. Since $\mathcal{F}$ is intersecting, any other interval under f is of the form $\{j, j-1, \ldots, j-(k-1)\}$ or $\{j+1, j+2, \ldots, j+k\}$ for some $j \in \{i, i+1, \ldots, i+k-1\}$. However, for each j we can get at most one of these two intervals since they are disjoint. Hence, there are at most $k-1$ such intervals and so there are at most k in total. Summing over all $n!$ bijections $[n] \to \mathbb{Z}_n$, we see that

$$N \leq kn!.$$

On the other hand, each $A \in \mathcal{F}$ is an interval under exactly $n(n-k)!k!$ bijections. So,

$$N = |\mathcal{F}|n(n-k)!k!,$$

and thus

$$|\mathcal{F}| \leq \frac{kn!}{n(n-k)!k!} = \frac{k}{n}\binom{n}{k} = \binom{n-1}{k-1}.$$

□

Remark 1 *The second proof goes under the name* Katona's circle method*. It can also be phrased probabilistically: take a uniformly random 'circular order' as above, and compute the expectation of the number of sets in $\mathcal{F}$ mapped to intervals.*

Remark 2 *For $k < n/2$, equality occurs only when $\mathcal{F}$ is a star. For $k = n/2$, this is not the case.*

Definition 2 (t-intersecting) *A set system $\mathcal{F} \subseteq 2^{[n]}$ is t-intersecting if $|A \cap B| \geq t$ for all $A, B \in \mathcal{F}$.*

Theorem 2 *Let $1 \leq t \leq k$. Then, there exists $n_0(k,t)$ such that for all $n \geq n_0$, if $\mathcal{F} \subseteq \binom{[n]}{k}$ is t-intersecting, then $|\mathcal{F}| \leq \binom{n-t}{k-t}$, with equality if and only if $\mathcal{F}$ is isomorphic to $\{A \subseteq [n] \mid [t] \subseteq A\}$.*

Proof Let $\mathcal{F}$ be maximal t-intersecting. Then there exist $A, B \in \mathcal{F}$ such that $|A \cap B| = t$. Assume there exists $C \in \mathcal{F}$ so that $A \cap B \not\subseteq C$. Then, every $D \in \mathcal{F}$ must have at least $t+1$ elements in $A \cup B \cup C$. Thus, for large nj,

$$|\mathcal{F}| \leq \binom{|A \cup B \cup C|}{t+1}\binom{n}{k-t-1} = O_{k,t}(n^{k-t-1}) < \binom{n-t}{k-t}.$$

□

What happens when n is not too large? This is answered by the Ahlswede-Khachatrian theorem.

Example 2 (Frankl sets) *Let $\mathcal{F}_i(n,k,t) = \{A \in \binom{[n]}{k} : |A \cap [1, t+2i]| \geq t+i\}$, for $0 \leq i \leq \frac{n-t}{2}$. Each $\mathcal{F}_i$ is t-intersecting. Note that the extremal example in Theorem 2 is $\mathcal{F}_0$.*

Theorem 3 (Ahlswede-Khachatrian) *For any n, k, t the maximum size of a t-intersecting subfamily of $\binom{[n]}{k}$ is achieved by some $\mathcal{F}_i(n,k,t)$. Every extremal set system isomorphic to some $\mathcal{F}_i$.*

To find the optimal i when given n, k, t, compare the sizes of consecutive $\mathcal{F}_i$. This gives a range for n as a function of k, t. In particular, $\mathcal{F}_0$ is the largest when $n \geq n_0(k,t) = (t+1)(k-t+1)$.

Remark 3 (Erdős Matching Problem) *How large can a set system $\mathcal{F} \subset \binom{[n]}{k}$ be without containing s disjoint sets? For $s = 2$, this is solved by Erdős-Ko-Rado. It is conjectured that the answer is $\max(\binom{n}{k} - \binom{n-s+1}{k}, \binom{ks-1}{k})$. For large n it is known that $\binom{n}{k} - \binom{n-s+1}{k}$ is the maximum.*

Let us now consider a different type of intersection property.

Theorem 4 (Bollobás' Two Families Theorem) *Let $\mathcal{A} = \{A_1, \ldots, A_m\}$ and $\mathcal{B} = \{B_1, \ldots, B_m\}$ be two set systems such that*

- *$A_i \cap B_i = \emptyset$, for all i, and*
- *$A_i \cap B_j \neq \emptyset$, for all $i \neq j$.*

Then,

$$\sum_{i=1}^{m} \binom{|A_i| + |B_i|}{|A_i|}^{-1} \leq 1.$$

Such pairs of set system are called *cross-intersecting*.

Corollary 1 *In the 'uniform' case, that is when $|A_i| = a$ and $|B_i| = b$ for all i, we obtain $m \leq \binom{a+b}{a}$.*

Corollary 2 *If $\mathcal{A}$ is an antichain on $[n]$ and $\mathcal{B} = \mathcal{A}^c = \{[n] \setminus A \mid A \in \mathcal{A}\}$, then $\mathcal{A}$ and $\mathcal{B}$ are cross-intersecting, so*

$$\sum_{k=0}^{n} \frac{|\mathcal{A} \cap \binom{[n]}{k}|}{\binom{[n]}{k}} \leq 1,$$

giving another proof of the LYM inequality.

Proof Without restriction, assume that all sets involved are subsets of $[n]$. For a permutation π of $[n]$, we write $A <_\pi B$ if $\max \pi(A) < \min \pi(B)$, where $\pi(S) := \{\pi(x) \mid x \in S\}$. Let $\pi \in S_n$ be chosen uniformly at random from all permutations of $[n]$. Then, for each i, since $A_i \cap B_i = \emptyset$, we have

$$\mathbb{P}(A_i <_\pi B_i) = \binom{|A_i| + |B_i|}{|A_i|}^{-1},$$

On the other hand, if $A_i <_\pi B_i$ then $A_j \not<_\pi B_j$ for $j \neq i$ (as $A_i \cap B_j$ and $A_j \cap B_i$ are both nonempty). So the events $\{A_i <_\pi B_i\}_{i \in [m]}$ are disjoint. Therefore,

$$1 \geq \mathbb{P}(\bigcup_{i \in [m]} \{A_i <_\pi B_i\}) = \sum_{i=1}^{m} \mathbb{P}(A_i <_\pi B_i) = \sum_{i=1}^{m} \binom{|A_i| + |B_i|}{|A_i|}^{-1}.$$

$\square$

Let us now consider an application of the Two Families Theorem. For tradition reasons we will use the language of "hypergraphs" rather than set systems.

Definition 3 (saturated) *Let H be a r-uniform hypergraph. We say that a r-uniform hypergraph G is H-*saturated* if G does not contain H as a subgraph, but after adding any missing edge to G, the resulting hypergraph will contain H as a subgraph.*

Definition 4 $\mathrm{sat}(n, H) := \min\{e(G) \mid |G| = n,\ G \textit{ is } H\textit{-saturated}\}$.

For instance, if $r = 2$ and $H = K_3$ then Mantel's Theorem tells us that the size of an H-free graph on n vertices is at most $\lfloor n^2/4 \rfloor$, and it is easy to see that $\mathrm{sat}(n, K_3) = n - 1$. In higher uniformities, very little is known about the the largest size of a $K_t^{(r)}$-free hypergraph. For $r = 3, t = 4$ this is the famous *Turan's tetrahedron problem.* In this light, it is surprising that Bollobás's theorem determines the saturation number precisely.

Theorem 5 *Let G be an r-uniform hypergraph on $[n]$, and suppose that adding any missing edge to G creates a copy of $K_{r+s}^{(r)}$. Then*

$$e(G) \geq \binom{n}{r} - \binom{n-s}{r}.$$

In particular, for $H = K_{r+s}^{(r)} := \binom{[r+s]}{r}$, we have $sat(n, H) = \binom{n}{r} - \binom{n-s}{r}$.

Proof Let $\mathcal{A} = \{A_1, \dots, A_m\} = \binom{[n]}{r} \setminus E(G)$ be the set of non-edges of G. For each i, there is an $(r+s)$-element set $K_i \supseteq A_i$ such that adding A_i to G creates a copy of $K_{r+s}^{(r)}$ with vertex set K_i. Let $B_i = [n] \setminus K_i$. Then

- $|A_i| = r$ and $|B_i| = n - r - s$ for each i;
- $A_i \cap B_i = \emptyset$ for each i;
- for distinct i, j, we have $A_i \cap B_j \neq \emptyset$ – else we would have $A_i \subseteq [n] \setminus B_j = K_j$, and so G would be missing two edges A_i, A_j from the complete r-graph with vertex set K_i, contradicting the definition of K_i.

So we can apply Corollary 1 to obtain

$$m \leq \binom{r + (n - r - s)}{r} = \binom{n-s}{r}.$$

$\square$

Remark 4 *The above bound is sharp, as witnessed by* $\{A \in \binom{[n]}{r} \mid A \cap [s] \neq \emptyset\}$.

Exercise 1 *Examine the proof to see if the extremal example is unique up to isomorphism.*