

NDMI113 – Extremal Combinatorics

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Lecture 3 - The Kruskal-Katona theorem.

Recall that the shift operator $C_{j \rightarrow i}$, for $i \neq j$, for a set $A \subseteq [n]$, is defined as

$$C_{j \rightarrow i}(A) = \begin{cases} (A \setminus \{j\}) \cup \{i\} & \text{if } j \in A \text{ and } i \notin A, \\ A & \text{otherwise.} \end{cases}$$

For a family $\mathcal{F} \subseteq 2^{[n]}$, it is defined as

$$C_{j \rightarrow i}(\mathcal{F}) = \{C_{j \rightarrow i}(A) \mid A \in \mathcal{F}\} \cup \{A \in \mathcal{F} \mid C_{j \rightarrow i}(A) \in \mathcal{F}\}.$$

Lemma 1 For $1 \leq i < j \leq n$ and $\mathcal{F} \subseteq \binom{[n]}{k}$, we have $|\partial(C_{j \rightarrow i}(\mathcal{F}))| \leq |\partial\mathcal{F}|$.

Proof Denote $\mathcal{G} = C_{j \rightarrow i}(\mathcal{F})$. Note that it suffices to show that for any $B^- \in \partial\mathcal{G} \setminus \partial\mathcal{F}$, we have

- i) $i \in B^-, j \notin B^-$ and
- ii) $(B^- \setminus \{i\}) \cup \{j\} \in \partial\mathcal{F} \setminus \partial\mathcal{G}$,

since then $C_{j \rightarrow i} : \partial\mathcal{G} \setminus \partial\mathcal{F} \rightarrow \partial\mathcal{F} \setminus \partial\mathcal{G}$ is injective.

Let $B^- \in \partial\mathcal{G} \setminus \partial\mathcal{F}$. By definition of the shadow, there exists $B \in \mathcal{G}$ and $x \in [n]$ so that $B = B^- \cup \{x\}$. Moreover, $B \notin \mathcal{F}$, as otherwise we would have $B^- \in \partial\mathcal{F}$. So, $B \in \mathcal{G} \setminus \mathcal{F} = C_{j \rightarrow i}(\mathcal{F}) \setminus \mathcal{F}$, which implies $i \in B$ and $j \notin B$. Let

$$A := (B \setminus \{i\}) \cup \{j\},$$

and note that $A \in \mathcal{F} \setminus \mathcal{G}$.

To see that i) holds, note first that $j \notin B \supseteq B^-$, so $j \notin B^-$. Furthermore, if $x \neq i$ then $i \in B \setminus \{x\} = B^-$. On the other hand, if $x = i$, we have

$$B^- = B \setminus \{i\} \subseteq A \in \mathcal{F},$$

a contradiction since $B^- \notin \partial\mathcal{F}$ by our assumption.

To prove ii) assuming i), we put $A^- := (B^- \setminus \{i\}) \cup \{j\}$. Then $A^- \subseteq A \in \mathcal{F}$, so $A^- \in \partial\mathcal{F}$, and we need to show that $A^- \notin \partial\mathcal{G}$. Suppose for the sake of contradiction that $A^- \in \partial\mathcal{G}$. Then, there exists $z \in [n]$ so that $A^- \cup \{z\} \in \mathcal{G}$. We consider two cases:

1. $z = i$: Then

$$A^- \cup \{z\} = (B^- \setminus \{i\}) \cup \{j\} \cup \{i\} = B^- \cup \{j\} \in \mathcal{G}.$$

Since $\{i, j\} \subset B^- \cup \{j\} \in C_{j \rightarrow i}(\mathcal{F})$ we must have $B^- \cup \{j\} \in \mathcal{F}$. This implies $B^- \in \partial\mathcal{F}$, a contradiction.

2. $z \neq i$: Note that $C_{j \rightarrow i}(\mathcal{G}) = \mathcal{G}$ since $\mathcal{G} = C_{j \rightarrow i}(\mathcal{F})$. So,

$$C_{j \rightarrow i}(A^- \cup \{z\}) = B^- \cup \{z\} \in \mathcal{G}.$$

But then $A^- \cup \{z\}$ and $C_{j \rightarrow i}(A^- \cup \{z\})$ are both in $\mathcal{G}$, so both are in $\mathcal{F}$. So, $B^- \cup \{z\} \in \mathcal{F}$, meaning $B^- \in \partial\mathcal{F}$ – again, a contradiction.

□

Definition 1 We say that a family $\mathcal{F} \subset 2^{[n]}$ is (left-)shifted if $C_{j \rightarrow i}(\mathcal{F}) = \mathcal{F}$ for all $1 \leq i < j \leq n$.

Corollary 1 Let $\mathcal{F} \subseteq \binom{[n]}{k}$. Then there exists a shifted family $\mathcal{A} \subseteq \binom{[n]}{k}$ such that $|\mathcal{A}| = |\mathcal{F}|$ and $|\partial\mathcal{A}| \leq |\partial\mathcal{F}|$.

Proof Define the function on the set systems

$$f(\mathcal{A}) := \sum_{A \in \mathcal{A}} \sum_{a \in A} a.$$

Then, for $i < j$ we see that $f(C_{j \rightarrow i}(\mathcal{A})) \leq f(\mathcal{A})$. In fact, applying a left-shift to $\mathcal{A}$ either leaves $\mathcal{A}$ unchanged, or strictly decreases $f(\mathcal{A})$. Hence, applying non-trivial left-shifts to $\mathcal{F}$ whenever we can, the process must terminate after finitely many steps. The resulting family $\mathcal{A}$ will be left-shifted, of size $|\mathcal{A}| = |\mathcal{F}|$, and, by Lemma 1, will satisfy $|\partial\mathcal{A}| \leq |\partial\mathcal{F}|$. □

Note that any initial segment of colex is shifted. However, the converse is not true – a shifted set system need not be isomorphic to the initial segment of colex of the same length. For example, initial segments of lex are also shifted and in general, not isomorphic to initial segments of colex. Thus, Corollary 1 is not enough for a proof of Kruskal-Katona.

Under a closer inspection, we notice that the mere existence of the colex ordering on $\binom{\mathbb{N}}{k}$ implies the following curious fact about the natural numbers.

Observation 1 (k -cascade form) Given $k \geq 1$, every integer $m \geq 1$ can be uniquely written as

$$m = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \cdots + \binom{a_s}{s},$$

where $a_k > a_{k-1} > \cdots > a_s \geq s \geq 1$.

Indeed, an inspection of the m -th smallest member of the k -colex readily gives the desired representation. For example, if $m = 9$ and $k = 3$, the corresponding set is 245. The initial segment of 3-colex of length 9 is therefore

$$\binom{[4]}{3} \cup \left(\binom{[3]}{2} + 5 \right) \cup \left(\binom{[2]}{1} + \{4, 5\} \right),$$

¹ whereby

$$9 = \binom{4}{3} + \binom{3}{2} + \binom{2}{1}.$$

¹Here, + stands for entering the elements into every member of the set, e.g. $\binom{[2]}{1} + \{4, 5\} = \{145, 245\}$.

Exercise 1 *Prove algebraically that the k -cascade form is well-defined.*

The cascade form of initial segments of colex behaves particularly nicely with respect to shadows.

Observation 2 *The initial segment $\mathcal{I}$ of k -colex of length*

$$|\mathcal{I}| = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \cdots + \binom{a_s}{s}$$

satisfies

$$|\partial\mathcal{I}| = \binom{a_k}{k-1} + \binom{a_{k-1}}{k-2} + \cdots + \binom{a_s}{s-1}$$

Note that the second expression may no longer represent the $(k-1)$ -cascade form of $|\partial\mathcal{I}|$ (although, it is quite close).

Theorem 1 (Kruskal-Katona restated) *Let $\mathcal{F} \subseteq \binom{[n]}{k}$ be of size (in k -cascade form)*

$$m = |\mathcal{F}| = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \cdots + \binom{a_s}{s}.$$

Then

$$|\partial\mathcal{F}| \geq \binom{a_k}{k-1} + \binom{a_{k-1}}{k-2} + \cdots + \binom{a_s}{s-1}.$$

Proof We proceed by double induction on k (external) and m (internal). For $k = 1$ and any m the statement holds trivially. Now let $k > 1$ and m be fixed. Assuming the induction hypothesis that the theorem statement holds for all (k', m') with either $[k' < k$ and arbitrary $m']$ or $[k' = k$ and $m' < m]$, we aim to prove it for any k -uniform set system of size m . So, let $\mathcal{F}$ be such a set system.

By Lemma 1 we may assume that $\mathcal{F}$ is left-shifted. Let us partition $\mathcal{F}$ into

$$\mathcal{F}_1 := \{A \in \mathcal{F} \mid 1 \in A\} \quad \text{and} \quad \mathcal{F}_0 := \mathcal{F} \setminus \mathcal{F}_1,$$

and define

$$\mathcal{F}_1^- := \{A \setminus \{1\} \mid A \in \mathcal{F}_1\} \subseteq \partial\mathcal{F}.$$

Note that, since the shifts $C_{i \rightarrow 1}$ do not change $\mathcal{F}$, for each $A \in \mathcal{F}_0$ and $A^- = A \setminus \{i\} \in \partial\mathcal{F}_0$ we have $A = A^- \cup \{1\} = C_{i \rightarrow 1}(A) \in \mathcal{F}_1$, and so $A^- \in \mathcal{F}_1^-$. Hence,

$$\partial\mathcal{F}_0 \subseteq \mathcal{F}_1^- \subseteq \partial\mathcal{F}_1,$$

and therefore

$$|\partial\mathcal{F}| = |\partial\mathcal{F}_1| = |\mathcal{F}_1^-| + |\partial\mathcal{F}_1^-|.$$

We claim that

$$|\mathcal{F}_1^-| \geq \binom{a_k - 1}{k-1} + \cdots + \binom{a_s - 1}{s-1}.$$

This would readily imply the induction step statement. Indeed, applying the induction hypothesis² to the $(k-1)$ -uniform family $\mathcal{F}_1^-$ we see that

$$|\partial\mathcal{F}_1^-| \geq \binom{a_k-1}{k-2} + \cdots + \binom{a_s-1}{s-2}$$

(and note that if necessary, we define $\binom{x}{-1} = 0$). Since $|\partial\mathcal{F}| = |\mathcal{F}_1^-| + |\partial\mathcal{F}_1^-|$, we obtain

$$\begin{aligned} |\partial\mathcal{F}| &\geq \left(\binom{a_k-1}{k-1} + \cdots + \binom{a_s-1}{s-1} \right) + \left(\binom{a_k-1}{k-2} + \cdots + \binom{a_s-1}{s-2} \right) \\ &= \left(\binom{a_k-1}{k-1} + \binom{a_k-1}{k-2} \right) + \cdots + \left(\binom{a_s-1}{s-1} + \binom{a_s-1}{s-2} \right) \\ &= \binom{a_k}{k-1} + \cdots + \binom{a_s}{s-1}, \end{aligned}$$

as desired.

It remains to prove the claim

$$|\mathcal{F}_1^-| \geq \binom{a_k-1}{k-1} + \cdots + \binom{a_s-1}{s-1}.$$

Suppose otherwise that

$$|\mathcal{F}_1^-| < \binom{a_k-1}{k-1} + \cdots + \binom{a_s-1}{s-1}.$$

Then

$$|\mathcal{F}_0| = m - |\mathcal{F}_1| > \binom{a_k-1}{k} + \binom{a_{k-1}-1}{k-1} + \cdots + \binom{a_s-1}{s}.$$

By the induction hypothesis³ applied to the k -uniform family $\mathcal{F}_0$ with $|\mathcal{F}_0| < m$ and the prior established fact that $\partial\mathcal{F}_0 \subseteq \mathcal{F}_1^-$, we conclude that

$$|\mathcal{F}_1^-| \geq |\partial\mathcal{F}_0| \geq \binom{a_k-1}{k-1} + \binom{a_{k-1}-1}{k-2} + \cdots + \binom{a_s-1}{s-1},$$

a contradiction. □

A more easy-to-use form of the Kruskal-Katona is due to Lovász. Recall, that for an arbitrary real number x and integer $k \geq 0$ the (generalized) binomial coefficient function $\binom{x}{k}$ is defined as

$$\binom{x}{k} = \frac{x(x-1)\cdots(x-k+1)}{k!}.$$

For a fixed k this function is strictly monotone increasing in x on $[k, \infty]$, and $\binom{k}{k} = 1$. Hence, any $y \geq 1$ can be uniquely expressed as $y = \binom{x}{k}$ with $x \geq k$.

Theorem 2 (Kruskal-Katona, Lovász form) *Let $\mathcal{F} \subseteq \binom{[n]}{k}$ be of size $\binom{x}{k}$, where $x \geq k$ is a real number. Then*

$$|\partial\mathcal{F}| \geq \binom{x}{k-1}.$$

The proof is analogous to the proof of Theorem 1.

²For this, the right hand side needs to be in $(k-1)$ -cascade form, which it is, strictly speaking, not, since we might have $s-1 = 0$. In that case we ‘shorten’ the cascade form by one term - note that we would estimate it against $\binom{a_s-1}{0-2} = 0$.

³A similar remark to the previous footnote applies here.