NDMI113 – Extremal Combinatorics

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Lecture 2 - LYM inequality, set shifting.

Sperner's Theorem tells us that the size of a Sperner family $\mathcal{F}$ on $[n]$ is bounded by $\binom{n}{\lfloor n/2 \rfloor}$. The following generalization of this theorem, known as the LYM inequality, was proven independently by Bollobás (1965), Lubell (1966), Meshalkin (1963), and Yamamoto (1954).

Theorem 1 (LYM Inequality) *If $\mathcal{F} \subseteq 2^{[n]}$ is Sperner, then*

$$\sum_{k=0}^{n} \frac{|\mathcal{F}_k|}{\binom{n}{k}} \leq 1,$$

where $\mathcal{F}_k := \mathcal{F} \cap \binom{[n]}{k}$, i.e. $\mathcal{F}_k$ is the subset of $\mathcal{F}$ consisting of precisely the sets of size k.

We shall give two proofs of the LYM inequality. The first one is due to Frankl, and is an ingenious application of the probabilistic method.

Proof Let $\mathcal{C}$ be chain of length $n+1$ picked uniformly at random (among the $n!$ choices). Let $\mathcal{C} = \{C_0, C_1, \dots, C_n\}$ where $\emptyset = C_0 \subsetneqq C_1 \subsetneqq \cdots \subsetneqq C_n = [n]$. We have

$$1 \geq \mathbb{P}[\mathcal{C} \cap \mathcal{F} \neq \emptyset] = \sum_{k=0}^{n} \mathbb{P}[\mathcal{C} \cap \mathcal{F}_k \neq \emptyset] = \sum_{k=0}^{n} \frac{|\mathcal{F}_k|}{\binom{n}{k}}.$$

Here the first equality comes from noting that the events $\mathcal{C} \cap \mathcal{F}_i \neq \emptyset$ and $\mathcal{C} \cap \mathcal{F}_j \neq \emptyset$ for $i \neq j$ are mutually exclusive, as a chain can intersect an antichain in at most one element. The second equality comes from

$$\mathbb{P}[\mathcal{C} \cap \mathcal{F}_k \neq \emptyset] = \frac{|\mathcal{F}_k|}{\binom{n}{k}},$$

since, by symmetry of $2^{[n]}$, the chain is equally likely to hit any given element of $\binom{[n]}{k}$. □

Our second proof of LYM requires the following fact known as the "Local LYM inequality".

Definition 1 *Let $\mathcal{F} \subseteq \binom{[n]}{k}$ be a uniform set system. The* shadow *of $\mathcal{F}$ is defined as*

$$\partial\mathcal{F} := \{B \in \binom{[n]}{k-1} : B \subseteq A, \text{ for some } A \in \mathcal{F}\}.$$

Theorem 2 (Local LYM) *For any uniform set system $\mathcal{F} \subseteq \binom{[n]}{k}$ we have*

$$\frac{|\partial\mathcal{F}|}{\binom{n}{k-1}} \geq \frac{|\mathcal{F}|}{\binom{n}{k}},$$

with equality if and only if $\mathcal{F} = \binom{[n]}{k}$ or $\mathcal{F} = \emptyset$.

Proof We argue by double counting the edges between $\mathcal{F}$ and $\partial\mathcal{F}$ in the inclusion graph (see Lecture 1). Each set in $\mathcal{F}$ is a superset of precisely k sets in $\partial\mathcal{F}$, and thus there are exactly $k \cdot |\mathcal{F}|$ edges between $\mathcal{F}$ and $\partial\mathcal{F}$. On the other hand, each set in $\partial\mathcal{F}$ is a subset of at most $n-(k-1)$ sets in $\mathcal{F}$ (it is a subset of exactly $n-(k-1)$ sets in $\binom{[n]}{k}$, some of which may not be in $\mathcal{F}$), so the edge set has cardinality at most $(n-k+1)\cdot|\partial\mathcal{F}|$. We obtain

$$|\partial\mathcal{F}| \geq |\mathcal{F}| \cdot \frac{k}{n-k+1} = |\mathcal{F}|\frac{\binom{n}{k-1}}{\binom{n}{k}}.$$

Equality occurs if for all $A \in \mathcal{F}$, $a \in A$, and $x \notin A$ we have $(A \setminus \{a\}) \cup \{x\} \in \mathcal{F}$, which can only happen if $\mathcal{F} = \binom{[n]}{k}$ or $\mathcal{F} = \emptyset$. $\square$

Proof [of the LYM inequality.] We may assume that $\mathcal{F} \neq \emptyset$. We first prove the following claim by downward induction. Define $\mathcal{G}_n = \mathcal{F}_n$ and for each $\ell < n$ define $\mathcal{G}_\ell = \mathcal{F}_\ell \cup \partial\mathcal{G}_{\ell+1}$. We claim that for all $\ell \leq n$ we have

$$\frac{|\mathcal{G}_\ell|}{\binom{n}{\ell}} \geq \sum_{k=\ell}^{n} \frac{|\mathcal{F}_k|}{\binom{n}{k}}.$$

The base case $\ell = n$ is trivial. Now we suppose the claim has been proven for $\ell+1$ and we shall prove it holds for ℓ. Notice first that $\mathcal{F}_\ell$ and $\partial\mathcal{G}_{\ell+1}$ are disjoint since each set in $\partial\mathcal{G}_{\ell+1}$ is contained in some element of $\mathcal{F}$, but $\mathcal{F}$ is an antichain. Therefore,

$$\frac{|\mathcal{G}_\ell|}{\binom{n}{\ell}} = \frac{|\mathcal{F}_\ell|}{\binom{n}{\ell}} + \frac{|\partial\mathcal{G}_{\ell+1}|}{\binom{n}{\ell}}.$$

Applying Local LYM yields

$$\frac{|\mathcal{G}_\ell|}{\binom{n}{\ell}} \geq \frac{|\mathcal{F}_\ell|}{\binom{n}{\ell}} + \frac{|\mathcal{G}_{\ell+1}|}{\binom{n}{\ell+1}}.$$

We now apply the induction hypothesis to the last term to get

$$\frac{|\mathcal{G}_\ell|}{\binom{n}{\ell}} \geq \frac{|\mathcal{F}_\ell|}{\binom{n}{\ell}} + \sum_{k=\ell+1}^{n} \frac{|\mathcal{F}_k|}{\binom{n}{k}} = \sum_{k=\ell}^{n} \frac{|\mathcal{F}_k|}{\binom{n}{k}}.$$

Thus, the claim is proven. Now consider the case $\ell = 0$ to get

$$1 = \frac{|\mathcal{G}_0|}{\binom{n}{0}} \geq \sum_{k=0}^{n} \frac{|\mathcal{F}_k|}{\binom{n}{k}},$$

as desired. $\square$

Observation 1 *The LYM inequality implies Sperner's Theorem. Indeed, we have*

$$1 \geq \sum_{k=0}^{n} \frac{|\mathcal{F}_k|}{\binom{n}{k}} \geq \sum_{k=0}^{n} \frac{|\mathcal{F}_k|}{\binom{n}{\lfloor n/2 \rfloor}} = \frac{|\mathcal{F}|}{\binom{n}{\lfloor n/2 \rfloor}}.$$

Furthermore, equality in Sperner implies equality in LYM and in the second inequality above. Hence $\mathcal{F} = \binom{[n]}{\lfloor n/2 \rfloor}$ *and* $\binom{[n]}{\lceil n/2 \rceil}$ *are the only equality cases for Sperner's theorem.*

Let us now discuss the shadows in more detail. In particular, for a k-uniform set system $\mathcal{F}$, how to minimize $|\partial\mathcal{F}|$, given $|\mathcal{F}|$. We may apply Local LYM to get

$$\frac{|\partial\mathcal{F}|}{\binom{n}{k-1}} \geq \frac{|\mathcal{F}|}{\binom{n}{k}}.$$

This bound is very good when $|\mathcal{F}|$ is close to $\binom{n}{k}$, but becomes really bad for small (e.g. constant) $|\mathcal{F}|$. So, what to do instead? For $|\mathcal{F}| = \binom{t}{k}$ for some $t \leq n$ it seems plausible that the family $\binom{[t]}{k}$ minimizes the shadow (e.g. this easy to verify for $k = 2$). If we believe this, then for general $|\mathcal{F}|$ we need to find a good way to interpolate between $\binom{[t]}{k}$ and $\binom{[t+1]}{k}$. This is achieved by the colexicographic ordering.

Definition 2 *The* colexicographic *(*colex*, for short) order on $\binom{\mathbb{N}}{k}$ is the total order in which*

$$A < B \text{ if } A \neq B \text{ and } \max(A \triangle B) \in B.$$

This can also be expressed as

$$A < B \text{ if } \sum_{i \in A} 2^i < \sum_{i \in B} 2^i.$$

Example 1 *Let $k = 3$. The first 10 elements in the colex order in uniformity k are (omitting parentheses):*

$$123, 124, 134, 234, 125, 135, 235, 145, 245, 345.$$

Remark 1 *Note that for $m = \binom{n}{k}$ the initial segment of length m of colex on $\binom{\mathbb{N}}{k}$ is precisely $\binom{[n]}{k}$. Thus, colex is also an ordering on $\binom{[n]}{k}$, which is consistent for a fixed k between different n.*

Remark 2 *Colex is similar to the better known* lexicographic order *(*lex*, for short), in which*

$$A < B \text{ if } A \neq B \text{ and } \min(A \triangle B) \in A.$$

This is just the familiar dictionary order. Note though, that for $k \geq 2$ lex, unlike colex, does not form a well-ordering of $\binom{\mathbb{N}}{k}$ of ordinal type ω. Also, lex on $\binom{[n]}{k}$ is not consistent for a fixed k between different n.

With this in hand, we can state the Kruskal-Katona Theorem on minimizing the shadow.

Theorem 3 (Kruskal-Katona) *Let $\mathcal{F} \subseteq \binom{[n]}{k}$, and let $\mathcal{A}$ be the family consisting of the first $|\mathcal{F}|$ elements of $\binom{n}{k}$ in colex order. Then*

$$|\partial\mathcal{F}| \geq |\partial\mathcal{A}|.$$

A major technique in the proof will be set shifting.

Definition 3 *For a set $A \subseteq [n]$ and distinct $i, j \in [n]$ we define the* shifting operator *$C_{j\to i}(A)$ as*

$$C_{j\to i}(A) = \begin{cases} (A \setminus \{j\}) \cup \{i\} & \text{if } j \in A \text{ and } i \notin A, \\ A & \text{otherwise.} \end{cases}$$

For a family $\mathcal{F}$, we define

$$C_{j\to i}(\mathcal{F}) = \{C_{j\to i}(A) \mid A \in \mathcal{F}\} \cup \{A \in \mathcal{F} \mid C_{j\to i}(A) \in \mathcal{F}\}.$$

For $i < j$, we call $C_{j\to i}$ a left-shift*, or a* left-compression.

Example 2 *Let* $n = 5, k = 3$ *and (omitting parentheses)* $\mathcal{F} = \{123, 134, 234, 235\}$. *Let* $i = 1$ *and* $j = 2$. *Then*

$$\begin{aligned} C_{j\to i}(123) &= 123, \\ C_{j\to i}(134) &= 134, \\ C_{j\to i}(234) &= 134 \in \mathcal{F}, \ \textit{and} \\ C_{j\to i}(235) &= 135 \notin \mathcal{F}. \end{aligned}$$

Therefore,

$$C_{j\to i}(\mathcal{F}) = \{123, 134, 234, 135\}.$$

Observation 2 *The following hold for any* $A \subseteq [n]$ *and* $\mathcal{F} \subseteq 2^{[n]}$*:*

1. $|C_{j\to i}(A)| = |A|$,
2. $|C_{j\to i}(\mathcal{F})| = |\mathcal{F}|$,
3. $C_{j\to i}(C_{j\to i}(\mathcal{F})) = C_{j\to i}(\mathcal{F})$.