NDMI113 – Extremal Combinatorics

Mykhaylo Tyomkyn

Lecture 10 - Lower bounds for weak saturation.

As we have seen before, upper bounds (constructions) for weak saturation numbers tend to be easy while lower bounds (impossibility proofs) are usually difficult. The following observation of Lovász/Kalai allows to tackle to lower bounds constructively. For simplicity let us deal with uniformity $r = 2$ throughout.

Theorem 1 *Let H be a graph, V a vector space over some field, and suppose there is a mapping $f : E(K_n) \to V$ such that for every isomorphic copy $H' \subseteq K_n$ of H the vectors $\{f_e = f(e) : e \in H'\}$ satisfy*

$$\sum_{e \in H'} \lambda_e f_e = 0, \tag{1}$$

with $\lambda_e \neq 0$ for ***all*** *$e \in H'$. Then*

$$wsat(n, H) \geq \dim \operatorname{span}\{f_e : e \in E(K_n)\}.$$

Proof Suppose G is a weakly H-saturated r-graph with n vertices and $wsat(n, H)$ many edges. Let $e_1, \ldots, e_m$ be a saturating edge sequence for G, and let $G_i = G + \{e_1, \ldots, e_i\}$ for each i, so that $G_0 = G$ and $G_m = K_n$. We now claim that for $i = m - 1, \ldots, 0$ we have

$$f(e_{i+1}) \in \operatorname{span}\{f_e : e \in E(G_i)\}.$$

Indeed, adding e_{i+1} to G_i creates, by definition, a new copy of H, let us call it H'. Then (1) implies that the vector $f(e_{i+1})$ is a linear combination of the vectors $\{f_e : e \in E(H') \setminus \{e_{i+1}\}\} \subseteq \{f_e : e \in E(G_i)\}$. Note that here we crucially used the fact that all coefficients in (1) are non-zero.

Therefore, we obtain

$$\operatorname{span}\{f_e : e \in E(G_i)\} = \operatorname{span}\{f_e : e \in E(G_{i+1})\},$$

and inductively

$$\operatorname{span}\{f_e : e \in E(G)\} = \operatorname{span}\{f_e : e \in E(G_0)\} = \operatorname{span}\{f_e : e \in E(G_m)\} = \operatorname{span}\{f_e : e \in E(K_n)\}.$$

It follows that

$$wsat(n, H) = |E(G)| \geq \dim \operatorname{span}\{f_e : e \in E(G)\} = \dim \operatorname{span}\{f_e : e \in E(K_n)\}.$$

□

If we want to apply Theorem 1, we need to construct a vector space V and a mapping f satisfying (1). The larger the dimension of V, the better the bound we obtain. Ideally, $\dim V$ should match the size of a constructed weakly H-saturated n-vertex graph G, in which case we would have determined $wsat(n, H)$ exactly.

This naturally leads to a discussion of matroids defined on graphs. To this end, let us first recall the definition(s) of a matroid.

Definition 1 (Matroid) *A* matroid *is a pair $\mathcal{M} = (E, \mathcal{I})$ where E is a finite set, called the set of* elements*, or the ground set, and $\mathcal{I} \subseteq 2^E$ is a set system, whose members are called* independent sets *and satisfy the following axioms*

(I1) $\emptyset \in \mathcal{I}$.

(I2) If $A \in \mathcal{I}$ and $B \subseteq A$ then $B \in \mathcal{I}$.

(I3) If $A, B \in \mathcal{I}$ and $|A| > |B|$ then there exists $x \in A \setminus B$ such that $B \cup \{x\} \in \mathcal{I}$.

Two main sources of examples of matroids are vector spaces and graphs.

Example 1 (Linear matroid) *Given a finite set E of vectors in some vector space V, set $\mathcal{I}$ to be the family of all linearly independent subsets of E. We know from Linear algebra that $\mathcal{I}$ satisfies (I1)-(I3). Often such a set of vectors presents itself as a set of rows (columns) of a matrix M. In such cases we speak of a* row matroid *(column matroid) of M.*

Example 2 (Graphic matroid) *Let G be a finite graph. Define $E = E(G)$ and $\mathcal{I}$ to be the set of all sub-forests of G. We know from Graph theory that $\mathcal{I}$ satisfies (I1)-(I3).*

Looking at both examples, one can quickly guess that there is a lot more to matroids than just independent sets. The following generalizes the notion of dimension of a vector space.

Definition 2 (Rank function) *Given a matroid $\mathcal{M} = (E, \mathcal{I})$ for every set $A \subseteq E$ its* rank *$r(A)$ is the size of the largest independent subset of A. The rank of $\mathcal{M}$ is defined as $r(\mathcal{M}) = r(E)$.*

The properties of the rank function implied by the axioms of independent sets are as follows.

(R1) $0 \leq r(A) \leq |A|$

(R2) If $A \subseteq B$ then $r(A) \leq r(B)$

(R3) $r(A \cup B) + r(A \cap B) \leq r(A) + r(B)$

It turns out that properties (R1)-(R3) of an integer-valued function on 2^E define a matroid uniquely, and thus can be taken as an alternative definition. To obtain $\mathcal{I}$ from r, set $\mathcal{I} = \{I \subseteq E : r(I) = |I|\}$. This is an example of a *cryptomorphism* between two different matroid axiomatizations.

There is also a generalization of the notion of a graph cycle.

Definition 3 (Circuit) *A circuit in a matroid $(E, \mathcal{I})$ is a set $C \subseteq E$ such that $C \notin \mathcal{I}$ but $C \setminus \{x\} \in \mathcal{I}$ for every $x \in C$.*

The family of circuits $\mathcal{C}$ has the following properties, which again describe the matroid uniquely.

(C1) $\emptyset \notin \mathcal{C}$.

(C2) $\mathcal{C}$ is a Sperner family.

(C3) For any distinct $C_1, C_2 \in \mathcal{C}$, if $e \in C_1 \cap C_2$, then there exists $C_3 \in \mathcal{C}$ such that $C_3 \subseteq C_1 \cup C_2 \setminus \{e\}$.

Exercise 1 *Give a reverse cryptomorphism. That is, reconstruct $\mathcal{I}$ from $\mathcal{C}$, deriving (I1)-(I3) from (C1)-(C3).*

This is only the tip of the iceberg: a matroid can be defined in a multitude of ways: using bases (maximal independent sets), flats (subspaces), hyperplanes, to name a few. All of these definitions are pairwise equivalent, via appropriate cryptomorphisms. As such, when speaking of a matroid, one usually writes $\mathcal{M}$ meaning *all* of the above: $E, \mathcal{I}, r, \mathcal{C}$ etc.

With matroids, Theorem 1 can be rephrased entirely in the language of set systems as follows.

Theorem 2 *Let H be a graph without isolated vertices, $\mathcal{M}$ a matroid with $E(\mathcal{M}) = E(K_n)$, and suppose that for every isomorphic copy $H' \subseteq K_n$ of H it holds that*

$$E(H') \in \mathcal{C}(\mathcal{M}).$$

Then

$$wsat(n, H) \geq r(\mathcal{M}).$$

If $wsat(n, H)$ can be exactly determined in this way, we say that H is *weak saturation matroidal* (*wsm*, for short). Let us now view some examples.

It is easy to see that $H = K_3$ is wsm. The underlying matroid is the graphic matroid of the complete graph K_n, where every triangle is a circuit and the rank is $n - 1$. The next case $H = K_4$ is already much more interesting.

Definition 4 (Count matroid) *A matroid $\mathcal{M}$ on $E = E(K_n)$ is a* count matroid *if there exist $k, \ell \in \mathbb{Z}$ such that (interpreting edge-sets as graphs without isolated vertices)*

$$G \in \mathcal{I} \iff \left(|E(G')| \leq k|V(G')| - \ell \textit{ for all } G' \subseteq G\right). \tag{2}$$

Graphs G satisfying (2) are called (k, ℓ)-tight.

Exercise 2 *What are the admissible values (k, ℓ) of count matroids?*

Example 3 *The count matroid $\mathcal{R}$ with $(k, \ell) = (2, 3)$ is a witness that K_4 is wsm. Indeed, K_4 is a circuit since removing any edge would make it $(2, 3)$-tight. Hence, $wsat(n, K_4) \geq r(\mathcal{R}) = 2n-3$, which matches the upper bound constructions.*

The matroid $\mathcal{R}$ is known as the *plane rigidity matroid.* The bases (maximal independent sets) of $\mathcal{R}$ are also known as *Laman graphs.* They have the following alternative characterization, involving the so-called *Henneberg extensions.*

- K_2 is a Laman graph

- Adding a new degree 2 vertex to a Laman graph results in a Laman graph
- Adding a new degree 3 vertex to a Laman graph and deleting one edge between its neighbours results in a Laman graph.

Moreover, every Laman graph can be obtained from K_2 via a sequence of Henneberg extensions.

Exercise 3 *Prove directly that every weakly K_4-saturated graph can be obtained from K_2 via a sequence of Henneberg extensions. Deduce that $wsat(n, K_4) = 2n - 3$.*

Our proof for larger cliques, using two-families theorem, can be recast as an instance of Theorem 2 (or Theorem 1, in fact). Does it give the same matroid for K_4? It turns out, it gives a different one!

Definition 5 (Hyperconnectivity matroid) *Let $\mathbf{p_1}, \ldots, \mathbf{p_n}$ be n points in $\mathbb{R}^2$ in general position (all coordinates are algebraically independent). The* plane hyperconnectivity matroid *$\mathcal{H}$ is the row matroid of the $\binom{n}{2} \times 2n$ matrix*

$$\begin{pmatrix} \mathbf{p_2} & -\mathbf{p_1} & 0 & \ldots & 0 & 0 \\ \mathbf{p_3} & 0 & -\mathbf{p_1} & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ \mathbf{p_n} & 0 & 0 & \ldots & 0 & -\mathbf{p_1} \\ 0 & \mathbf{p_3} & -\mathbf{p_2} & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & \mathbf{p_n} & -\mathbf{p_{n-1}} \end{pmatrix}$$

Exercise 4 *Show that K_4 and $K_{3,3}$ are $\mathcal{H}$-circuits and that $r(\mathcal{H}) = 2n - 3$.*

Note that $K_{3,3}$ is $(2,3)$-tight, and therefore $\mathcal{R}$-independent yet an $\mathcal{H}$-circuit. Hence $\mathcal{H} \neq \mathcal{R}$.

In this light, we should ask if every graph is wsm. This was answered in the negative in 2025.

Theorem 3 (Terekhov-Zhukovskii '25) *If H is wsm then its limiting constant c_H is an integer.*

Graphs whose limiting constant is not integral are abound.

Exercise 5 *Let H be the Moser spindle.*

1. *Construct a graph showing that $wsat(n, H) \leq 5n/3 + o(n)$.*
2. *Show that a weakly H-saturated graph cannot have two adjacent vertices of degree 2. Deduce that $wsat(n, H) \geq 6n/5 + o(n)$.*

Lastly, let us remark that weak saturation equally makes sense on *host graphs* other than the complete graph: For $G \subseteq F$ we call G weakly H-saturated in F if the edges in $E(F) \setminus E(G)$ can be ordered such that adding each new edge creates a new copy of H. It turns out that determining the weak saturation numbers in this generality is a fairly hopeless task even when H is the triangle.

Theorem 4 (Tancer-T. '25) *Given as an input a graph F on n vertices, it is NP-hard to decide whether $wsat(F, K_3) = n - 1$.*