On the Arrangement of Hyperplanes Determined by $n$ Points

Michal Opler, Pavel Valtr, Tung Anh Vu
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Given:

- an arrangement of $n$ hyperplanes in $\mathbb{R}_d$. 

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How many cells are there? Assuming general position.

\[
\Phi_d(n) = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{d}
\]
What is known? What are we doing?

Given:

- an arrangement of $n$ hyperplanes in $\mathbb{R}^d$
- points $P = \{p_1, \ldots, p_n\} \subseteq \mathbb{R}^d$, 

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How many cells are there? Assuming even more general position.
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How many cells are there? Assuming even more general position.

$$f_d(n) = ?$$
Our approach

- Consider an arrangement $\mathcal{A}$ of hyperplanes in $\mathbb{R}^d$ (in sufficiently general position).
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Theorem (Zaslavsky 1975)

\[ f_d(n) = |\chi_{\mathcal{A}}(-1)|. \]
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$\chi_{\mathcal{A}} = \text{characteristic polynomial of } \mathcal{A}$.

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Theorem (Whitney 1932)

$$\chi_{\mathcal{A}}(t) = \sum_{\mathcal{B} \subseteq \mathcal{A}} (-1)^{|\mathcal{B}|} t^{d-\text{rank}(\mathcal{B})} = \sum_{\mathcal{B} \subseteq \mathcal{A}} (-1)^{|\mathcal{B}|} t^{\text{dim}(\bigcap \mathcal{B})}.$$  

Def. $\mathcal{B}$ is central $\iff \bigcap_{H \in \mathcal{B}} H \neq \emptyset$.  

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Consider an arrangement \( \mathcal{A} \) of hyperplanes in \( \mathbb{R}^d \) (in sufficiently general position).

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\]

**Def.** \( \mathcal{B} \) is central \( \iff \bigcap_{H \in \mathcal{B}} H \neq \emptyset \).

\( \Rightarrow \) just determine \( \chi_{\mathcal{A}} \).
Our results

- algorithmic method to compute $f_d(n)$ as a polynomial in $n$ for fixed $d$, 

$$f_d(n) = \frac{d!}{d+1} \cdot n^{d^2} - \frac{d^3}{2} \cdot \left( \frac{d!}{d+1} \right) \cdot n^{d^2-1} + O(n^{d^2-2})$$

- the first $d-1$ coefficients of $\Phi_d((n^d))$ and $f_d(n)$ are equal.

$\Phi_d((n^d))$: number of cells in an arrangement of $(n^d)$ hyperplanes in general position.
Our results

- **algorithmic method** to compute $f_d(n)$ as a polynomial in $n$ for fixed $d$,
- determine $f_d(n)$ precisely for $d \leq 6$, 

$\quad f_d(n) = (d!)^{d+1} \cdot n^{d^2} - d^3 + O(n^{d^2-2})$, 

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- algorithmic method to compute $f_d(n)$ as a polynomial in $n$ for fixed $d$,
- determine $f_d(n)$ precisely for $d \leq 6$,
- $f_d(n)$ is a degree $d^2$ polynomial, and

$$f_d(n) = \frac{1}{(d!)^{d+1}} \cdot n^{d^2} + \frac{d^2 - d^3}{2 \cdot (d!)^{d+1}} \cdot n^{d^2-1} + O(n^{d^2-2}),$$
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- the first $d - 1$ coefficients of $\Phi_d \left( \binom{n}{d} \right)$ and $f_d(n)$ are equal.
  - $\Phi_d \left( \binom{n}{d} \right)$: number of cells in an arrangement of $\binom{n}{d}$ hyperplanes in general position.
$d = 2$

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\[
\left(n \cdot \left(n - 2\right) \right) \cdot \left(n - 1\right)
\]

\(\text{C}\)

\(\text{p}\)
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- The bottommost point of $C$ exists but is not a point of $P$.  

\[ n \cdot (n - 2) \cdot \left( \frac{n^2}{2} \right) \cdot \left( n - 2 \right) \cdot \left( \frac{n^2}{2} \right) \]
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- The bottommost point of \( C \) exists but is not a point of \( P \).
  \[ \frac{1}{2} \cdot \binom{n}{2} \cdot \left( \binom{n - 2}{2} \right) \]

- \( C \) is unbounded from below.
  \[ \binom{n}{2} + 1 \]
From $d = 2$ to $d = 3$

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Open problems

- Find a closed formula for $f_d(n)$.

- How many $k$-faces are there in a hyperplane arrangement (in sufficiently general position)?

- Improve the running time of our algorithm for determining the polynomial $f_d(n)$ when $d = 7$.

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