On the Arrangement of Hyperplanes Determined by n Points

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CTU IN PRAGUE

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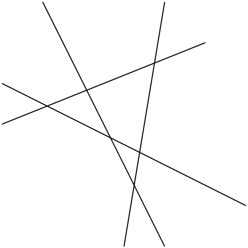
Given:

- an arrangement of n hyperplanes in $\mathbb{R}_d$.



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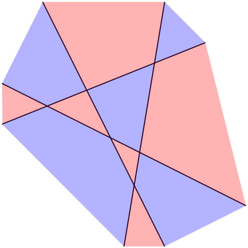

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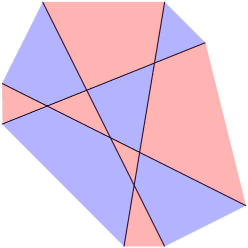
$$\Phi_d(n) = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{d}$$



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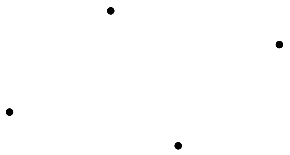

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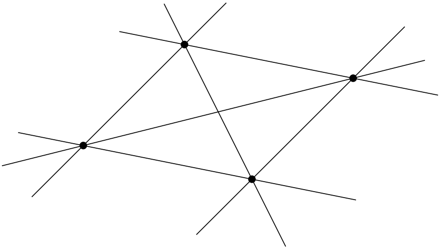

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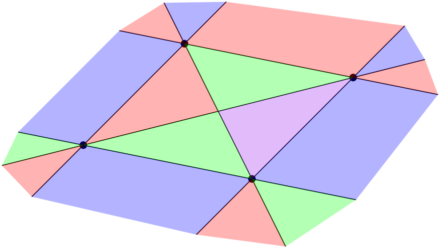

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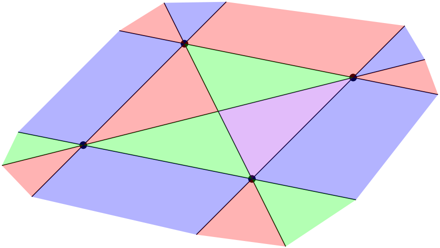
$$f_d(n) = ?$$



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$\Rightarrow$ just determine $\chi_{\mathcal{A}}$.

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- the first $d-1$ coefficients of $\Phi_d\left(\binom{n}{d}\right)$ and $f_d(n)$ are equal.
  - $\Phi_d\left(\binom{n}{d}\right)$: number of cells in an arrangement of $\binom{n}{d}$ hyperplanes in general position.

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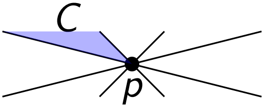

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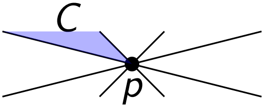

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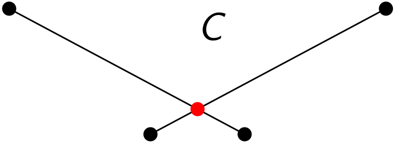

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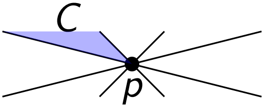

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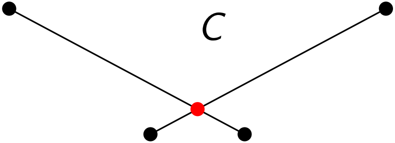

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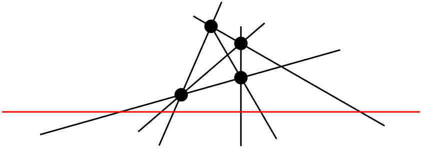

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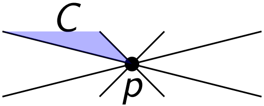

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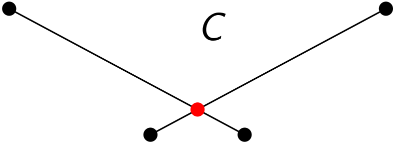
$\frac{1}{2} \cdot \binom{n}{2} \cdot \binom{n-2}{2}$



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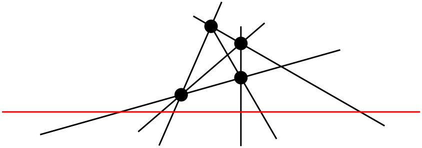
$\binom{n}{2} + 1$



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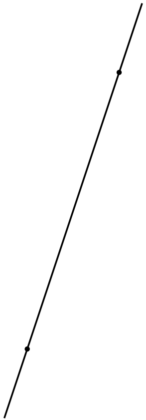

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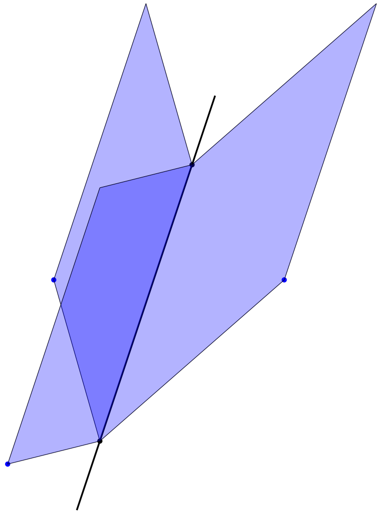

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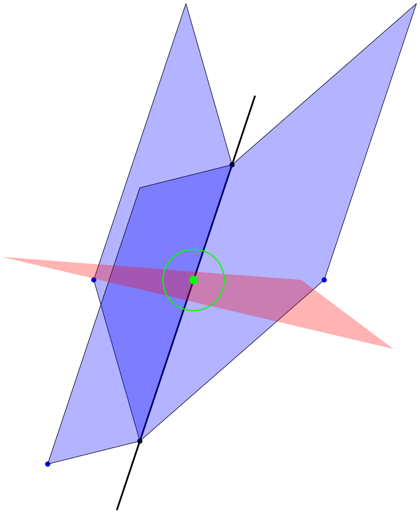

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Kamak workshop

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