Mathematical analysis I — Tutorial 1

http://kam.mff.cuni.cz/~tereza/teaching.html

Problem 1: Decide whether the following propositions are true and write their negations:

- Nine is a prime and nine is odd.
- Nine is a prime or nine is odd.
- If nine is a prime, then nine is odd.
- Nine is a prime if and only if nine is odd.

Problem 2: Decide whether the following propositions are true and write their negations:

1. $\forall x \in \mathbb{Z}\, \exists y \in \mathbb{Z}\, \forall z \in \mathbb{Z} : z > x \Rightarrow z > y$
2. $\exists y \in \mathbb{Z}\, \forall x \in \mathbb{Z}\, \forall z \in \mathbb{Z} : z > x \Rightarrow z > y$

What if we quantify over natural numbers instead of integers?

Problem 3: Is the following proposition or its negation true for $f(x) = \sin x$?

$$\forall \varepsilon > 0\ \exists K > 0\ \forall x : x > K \Rightarrow |f(x)| < \varepsilon.$$

Problem 4: Which of the following propositions implies the other one?

A $\forall x\ \exists K > 0 : |f(x+1) - f(x)| \leq K$

B $\exists K > 0\ \forall x : |f(x+1) - f(x)| \leq K$

Problem 5: Show that for any two reals x and y, such that $x > 0$, it holds that $(y \leq x) \wedge (-x \leq y) \Leftrightarrow |y| \leq x$.

Problem 6: Prove by mathematical induction:

$$\sum_{i=1}^{n}(2i-1) = n^2.$$

Problem 7: Prove inequality between arithmetic and geometric mean.

For non-negative $x_1, \dots, x_n$ it holds that: $\sqrt[n]{x_1 x_2 \cdots x_n} \leq \dfrac{x_1 + x_2 + \cdots + x_n}{n}$

a) For $n = 2$.

b) For $n = 2^k$ by induction on k.

Problem 8: Prove by strong induction that every natural number $n \geq 2$ is divisible by a prime.

Problem 9: Using the previous exercise, prove that there are infinitely many primes. (By contradiction.)

Mathematical analysis I — Homework 1

Due: 15:40, 10.10.

Write your solution of each problem on a separate sheet of paper. One part will be marked for credit.

Problem 1: Prove by induction that for every $n \in \mathbb{N}$, $10^n - 4$ is divisible by 6.

Problem 2: Prove by contradiction that for every positive rational number x there exists a positive rational number y such that $y < x$.

Challenge (not a part of the homework): Can you also show that for every positive **real** number x there exists a positive rational number y such that $y < x$?

Problem 3: Prove that for every natural number n, any two positive reals x and y satisfy that $x < y$ if and only if $x^n < y^n$. (Hint: Use mathematical induction to prove that $x < y \Rightarrow x^n < y^n$. Then show by contrapositive that $x^n < y^n \Rightarrow x < y$.) Do x and y have to be positive for the statement to be true?