Mathematical analysis I — Tutorial 4

http://kam.mff.cuni.cz/~tereza/teaching.html

Problem 1: Decide whether the following sequences are monotone. If yes, are they increasing, decreasing, non-increasing or non-decreasing?

a) $(2^{-n})_{n=1}^{\infty}$ decreasing

b) $(2n + (-1)^n)_{n=1}^{\infty}$ non-decreasing

c) $(\sin n)_{n=1}^{\infty}$ not monotone

d) $\left(\frac{1}{1+n^2}\right)_{n=1}^{\infty}$ decreasing

e) $\left(\frac{n+1}{n+2}\right)_{n=1}^{\infty}$ increasing

f) $(\sqrt{n+1} - \sqrt{n})_{n=1}^{\infty} = \left(\frac{1}{\sqrt{n+1}+\sqrt{n}}\right)_{n=1}^{\infty}$ decreasing

Problem 2: A sequence (a_n) is known to be increasing.

a) Might it have an upper bound? YES e.g. $(-\frac{1}{n})$

b) Might it have a lower bound? YES — // —

c) Must it have an upper bound? NO e.g. (n)

d) Must is have a lower bound? YES a_1 is a lower bound

Give a numerical example to illustrate each possibility or impossibility.

Problem 3: If a sequence is not bounded above, must it contain

a) a positive term, YES, by definition 0 is not upper bound, so $\exists n : a_n > 0$

b) an infinite number of positive terms? YES, by contradiction: assume (a_n) has finitely many positive terms. Then maximum of them is an upper bound. $\not\Sigma$

Problem 4: Think of examples to show that:

a) an increasing sequence need not tend to infinity; $(-\frac{1}{n})$

b) a sequence that tends to infinity need not be increasing; $(n + (-1)^n)$

c) a sequence with no upper bound need not tend to infinity. $((-1)^n n)$

Problem 5: Justify that the following properties do not imply that a sequence tends to zero (i.e. its limit is zero).

a) A sequence in which each term is strictly less than its predecessor. $(a_n) \to -\infty$

b) A sequence in which each term is strictly less than its predecessor while remaining positive. $(b_n) \to 1$

c) A sequence in which, for sufficiently large n, each term is less than some small positive number. $(c_n) \to -\frac{1}{10}$

d) A sequence with arbitrarily small terms. (d_n) has no limit

Use the following sequences for your arguments.

$$a_n = 3 - n, \qquad b_n = \frac{n+1}{n}, \qquad c_n = -\frac{1}{10}, \qquad d_n = \begin{cases} 1 \text{ for } n \text{ odd}, \\ \frac{1}{2^n} \text{ for } n \text{ even}. \end{cases}$$

Problem 6: For the sequence $a_n = 1 + \frac{1}{\sqrt{n}}$, find n_0 such that for every $n \geq n_0$

a) $|a_n - 1| < 0,1$ $n_0 > 100$ e.g. $n_0 = 101$

a) $|a_n - 1| < 0,01$ $n_0 > 10000 = 100^2$

Problem 7: Prove that the sequence $a_n = \frac{(-1)^n}{n}$ does not converge to 2.

for $\varepsilon = \frac{1}{2}$ $\forall n : \left| \frac{(-1)^n}{n} - 2 \right| > \varepsilon$

Problem 8

a) $\lim\limits_{n\to\infty} \frac{1}{n} = 0$

- $n \geq n_0 \Rightarrow \frac{1}{n_0} \geq \frac{1}{n}$ $\quad (*)$
- $\left|\frac{1}{n_0} - 0\right| < \varepsilon \iff \frac{1}{n_0} < \varepsilon \iff \frac{1}{\varepsilon} < n_0$
  - this holds for instance for $n_0 = \lceil \frac{1}{\varepsilon} \rceil + 1$
- Thus, by $(*)$ $\forall n \geq \lceil \frac{1}{\varepsilon} \rceil + 1$, it holds that $\left|\frac{1}{n} - 0\right| < \left(\left|\frac{1}{n_0} - 0\right| <\right) \varepsilon$

b) $\lim\limits_{n\to\infty} \log n = \infty$

- We need to show: $\forall K\ \exists n_0\ \forall n \geq n_0$ $\log n > K$
- $n > n_0 \Rightarrow \log n > \log n_0$, so if $\log n_0 > K$,
- ~~so~~ $\forall n > n_0 : \log n > K$
- $\log n_0 > K \iff n_0 > 10^K$
  - this holds for instance for $n_0 = \lceil 10^K \rceil + 1$

Thus, $\forall n \geq \lceil 10^K \rceil + 1$ satisfies $\log n > K$

c) $\lim\limits_{n\to\infty} \frac{1}{1+n^2} = 0$: $\forall n$: $0 \leq \frac{1}{1+n^2} \leq \frac{1}{n}$

Thus, by a) and Sandwich theorem limit of $\left(\frac{1}{1+n^2}\right)$ is 0.

d) has no limit, see the theorem about limit of a subsequence.

e) — // —

f) $\lim\limits_{n\to\infty} (-1)^{n!} = 1$ $\quad$ $n!$ is ~~odd~~ even for $n \geq 2$, so the sequence is $-1, 1, 1, 1, 1, \dots$

g) $\lim\limits_{n\to\infty} \frac{(-1)^n}{n} = 0$

- $-\frac{1}{n} \leq \frac{(-1)^n}{n} \leq \frac{1}{n}$
- $\lim\limits_{n\to\infty} \frac{1}{n} = 0$, $\lim\limits_{n\to\infty} -\frac{1}{n} \overset{\text{lim arith.}}{=} -1 \cdot \lim\limits_{n\to\infty} \frac{1}{n} = -1 \cdot 0 = 0$

So, by Sandwich theorem, limit of $\frac{(-1)^n}{n}$ is 0

h) $\lim\limits_{n\to\infty} \cos\left(\frac{n\pi}{4}\right)$ does not exist, consider subsequences

$$\left(\cos \frac{8k\pi}{4}\right)_{k=1}^{\infty} = (1)_{k=1}^{\infty}$$

$$\left(\cos\left((8k+4)\frac{\pi}{4}\right)\right)_{k=1}^{\infty} = (-1)_{k=1}^{\infty}$$

i) $\lim\limits_{n\to\infty} \dfrac{2n+1}{3n-2} = \lim\limits_{n\to\infty} \dfrac{n\left(2+\frac{1}{n}\right)}{n\left(3-\frac{2}{n}\right)} = \lim\limits_{n\to\infty} \dfrac{2+\frac{1}{n}}{3-\frac{2}{n}}$

$$\overset{\text{Arithm. of Limits}}{=} \frac{\lim\limits_{n\to\infty} 2 + \lim\limits_{n\to\infty} \frac{1}{n}}{\lim\limits_{n\to\infty} 3 - \lim\limits_{n\to\infty} 2 \cdot \lim\limits_{n\to\infty} \frac{1}{n}} = \frac{2+0}{3-2\cdot 0} = \underline{\underline{\frac{2}{3}}}$$

Bonus

$$\lim_{n\to\infty} \sqrt{n+1} - \sqrt{n} = \lim_{n\to\infty} \left(\sqrt{n+1} - \sqrt{n}\right) \cdot \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}$$

$$= \lim_{n\to\infty} \frac{n+1+n}{\sqrt{n+1}+\sqrt{n}} = \lim_{n\to\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}} = \lim_{n\to\infty} \frac{1}{\sqrt{n}} \cdot \frac{1}{\sqrt{1+\frac{1}{n}}+\sqrt{1}}$$

$$\overset{\text{Arithm. of Lim}}{=} \lim_{n\to\infty} \frac{1}{\sqrt{n}} \cdot \cancel{\lim} \frac{\cancel{4}\lim\limits_{n\to\infty} 1}{\lim\limits_{n\to\infty} \sqrt{1+\frac{1}{n}} + \lim\limits_{n\to\infty} \sqrt{1}} \qquad (*)$$

- $\lim\limits_{n\to\infty} \sqrt{1+\frac{1}{n}} = 1$, since ~~by Problem~~ $\lim\limits_{n\to\infty} 1+\frac{1}{n} \overset{\text{Ar. of Lim.}}{=} 1 + \lim\limits_{n\to\infty} \frac{1}{n} = 1+0 = 1$

and by problem 3 from HW, $\lim\limits_{n\to\infty} \sqrt{a_n} = \sqrt{\lim\limits_{n\to\infty} a_n}$ ($\cancel{4}$ if $(a_n) \geq 0$ and converges)

- $\lim\limits_{n\to\infty} \dfrac{1}{\sqrt{n}} = \lim\limits_{n\to\infty} \sqrt{\dfrac{1}{n}} = 0$, also by Problem 3

So, $(*) = 0 \cdot \dfrac{1}{1+1} = \underline{\underline{0}}$