Solution of the test from 14. 1.

$\boxed{1.}$ $\lim\limits_{n\to\infty} \sqrt[2n]{3^n + (2+\sin n)^n + (-1)^n} + \frac{n+2}{3n-(3^{-n})} \overset{AL}{=}$

$$= \underbrace{\lim_{n\to\infty} \sqrt[2n]{3^n+(2+\sin n)^n+(-1)^n}}_{=A} + \underbrace{\lim_{n\to\infty} \frac{n+2}{3n-(3^{-n})}}_{=B}$$

$$B = \lim_{n\to\infty} \frac{\cancel{n}\left(1+\frac{2}{n}\right)}{\cancel{n}\left(3-\frac{1}{n\cdot 3^n}\right)} \overset{AL}{=} \frac{1+\lim\limits_{n\to\infty}\frac{2}{n}}{3-\lim\limits_{n\to\infty}\frac{1}{n3^n}} = \frac{1+0}{3-0} = \frac{1}{3}$$

A: $\forall n:$ $(-1)^n \leq 3^n$

$(2+\sin n)^n \leq 3^n$

$(2+\sin n)^n + (-1)^n \geq 0$

Thus, $\sqrt[2n]{3^n} \leq \sqrt[2n]{3^n+(2+\sin n)^n+(-1)^n} \leq \sqrt[2n]{3^n+3^n+3^n}$

$\lim\limits_{n\to\infty} \sqrt[2n]{3^n} = \sqrt{3}$, $\lim\limits_{n\to\infty} \sqrt[2n]{3^n+3^n+3^n} = \lim\limits_{n\to\infty} \sqrt[2n]{3}\,\sqrt[2n]{3^n}$

$$\overset{AL}{=} \underbrace{\lim_{n\to\infty}\sqrt[2n]{3}}_{\text{known limit}} \cdot \lim_{n\to\infty}\sqrt[2n]{3^n} = 1\cdot\sqrt{3}$$

So, ~~A+B~~ by sandwich theorem, $A=\sqrt{3}$, so

$$A+B = \sqrt{3}+\frac{1}{3}$$

$\boxed{2.}$ $\boxed{\text{absolute convergence:}}$ $\left|\frac{(-1)^n}{n-\sqrt{n}} + \left(-\frac{1}{n}\right)^n\right| = \frac{1}{n-\sqrt{n}} + \frac{1}{n^n}$ $(n\geq 2)$

$\frac{1}{n-\sqrt{n}} + \frac{1}{n^n} > \frac{1}{n-\sqrt{n}} > \frac{1}{n}$, so by comparison criterion,

since $\sum \frac{1}{n}$ diverges, $\sum\left(\frac{1}{n-\sqrt{n}}+\frac{1}{n^n}\right)$ diverges.

Conclusion: given series does not converge absolutely

$\boxed{\text{conditional convergence}}$ If $\sum\limits_{n=2}^{\infty} \frac{(-1)^n}{n-\sqrt{n}}$ and $\sum\limits_{n=2}^{\infty} \left(-\frac{1}{n}\right)^n$ converge,

then $\sum\limits_{n=2}^{\infty} \frac{(-1)^n}{n-\sqrt{n}} + \left(-\frac{1}{n}\right)^n$ also converges.

- $\sum\limits_{n=2}^{\infty} \frac{(-1)^n}{n-\sqrt{n}}$ converges by Leibniz criterion, since

  $\lim\limits_{n\to\infty} \frac{1}{n-\sqrt{n}} = 0$ and $\left(\frac{1}{n-\sqrt{n}}\right)_{n=2}^{\infty}$ is

  non-increasing: we will show: $\frac{1}{n-\sqrt{n}} \geq \frac{1}{(n+1)-\sqrt{n+1}}$

  $$\Longleftrightarrow \quad (n+1) - \sqrt{n+1} \geq n - \sqrt{n}$$

  $$\Longleftrightarrow \quad 1 \geq \sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1}+\sqrt{n}}$$

  since $\sqrt{n}, \sqrt{n+1} \geq 1$, the last inequality holds, so the claim holds as well.

- $\sum\limits_{n=2}^{\infty} \left(-\frac{1}{n}\right)^n$ converges, because it converges absolutely

  by root rule: $\lim\limits_{n\to\infty} \sqrt[n]{\frac{1}{n^n}} = \lim\limits_{n\to\infty} \frac{1}{n} = 0 < 1$

Conclusion: given series converges (conditionally).

$\boxed{3.}$ $$\lim_{x\to 0} \frac{x^2\,(e^x-1)}{\sin x \cdot \cos 3x - \sin x} = \lim_{x\to 0} \frac{x}{\sin x} \cdot \frac{(e^x-1)}{x} \cdot \frac{x^2}{\cos 3x - 1}$$

$$\overset{AL}{=} \underbrace{\lim_{x\to 0} \frac{x}{\sin x}}_{=1} \cdot \underbrace{\lim_{x\to 0} \frac{e^x-1}{x}}_{=1} \cdot \lim_{x\to 0} \frac{x^2}{\cos 3x - 1} = \lim_{x\to 0} \frac{x^2}{\cos 3x - 1}$$

known limits

$$\underset{\text{"}\frac{0}{0}\text{"}}{\overset{L'H}{=}} \lim_{x\to 0} \frac{2x}{-3\sin 3x} \underset{\text{"}\frac{0}{0}\text{"}}{\overset{L'H}{=}} \lim_{x\to 0} \frac{2}{-3\cdot 3 \cdot \cos 3x} \overset{\boxed{\cos 3\cdot 0 = 1}}{=} \frac{2}{-9}$$

$\boxed{4.}$ $D_f = \mathbb{R}$ since $x^2+3 > 0 \ \forall x$, so $\sqrt{x^3+3}$ is always <u>defined</u> and <u>positive</u>

$$\left(\sqrt{x^2+3}\right)^{\sin x} = e^{\frac{\sin x}{2}\cdot \ln(x^2+3)}$$

$$\left(e^{\frac{\sin x}{2}\cdot \ln(x^2+3)}\right)' = e^{\frac{\sin x}{2}\ln(x^2+3)}\left(\frac{1}{x^2+3}\cdot \cancel{2}x\cdot\frac{\sin x}{\cancel{2}} + \frac{\cos x}{2}\cdot \ln(x^2+3)\right).$$

exists for every $x \in \mathbb{R}$