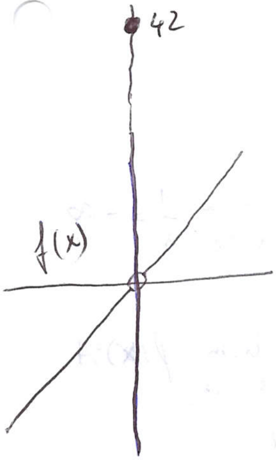
Remark: $\lim_{x \to a} f(x)$ does not depend on $f(a)$, f might ~~not~~ even be undefined in a



Examples

- $f(x) \begin{cases} x & x \neq 0 \\ 42 & x = 0 \end{cases}$. $\lim_{x \to 0} f(x) = 0$

  $\forall \varepsilon > 0 \ \exists \delta > 0 : |x| < \delta \Rightarrow |f(x)| < \varepsilon$

  Enough to choose $\delta := \varepsilon$.

  . $\lim_{x \to \infty} f(x) = \infty$: $\forall \varepsilon > 0 \ \exists \delta > 0 : x > \frac{1}{\delta} \Rightarrow f(x) > \frac{1}{\varepsilon}$

  Again, take $\delta := \varepsilon$

  . $\lim_{x \to -\infty} f(x) = -\infty$: $\forall \varepsilon > 0 \ \exists \delta > 0 : x < -\frac{1}{\delta} \Rightarrow f(x) < -\frac{1}{\varepsilon}$

- $f: \mathbb{Q} \to \mathbb{Q}$, $f(p/q) = 1/q$, where p/q is irreducible

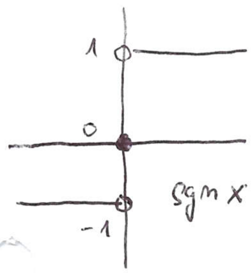
  $\lim_{x \to 0} f(x) = 0$ (guess) : $\forall \varepsilon > 0 \ \exists \delta > 0 : |x| < \delta \Rightarrow |f(x)| < \varepsilon$

  Given ε, find n s.t. $\frac{1}{n} < \varepsilon$, take $\delta = \frac{1}{n}$

- $\lim_{x \to 0} \frac{e^x - 1}{x} \underset{\text{guess}}{=} 1$: $\forall \varepsilon > 0 \ \exists \delta > 0 : |x| < \delta \Rightarrow \left| \frac{e^x - 1}{x} - 1 \right| < \varepsilon$

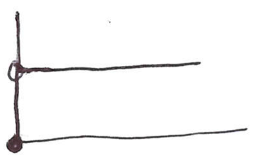
  $\boxed{\text{for } |x| < \frac{1}{2}, \ x \neq 0}$ $\left| \frac{e^x - 1}{x} - 1 \right| \leq \sum_{n=1}^{\infty} \frac{|x|^n}{(n+1)!} < \sum_{n=1}^{\infty} |x|^n = \frac{|x|}{1 - |x|} < 2|x|$

  So, taking $\delta := \frac{\varepsilon}{2}$ works.



- $\lim_{x \to 0} \operatorname{sgn} x$. $\lim_{x \to 0} \operatorname{sgn} x$ does not exist

  $\lim_{x \to a} \operatorname{sgn} x = 1$ for $a > 0$, -1 for $a < 0$



How to formally express that, nevertheless, $\operatorname{sgn} x$ tends to 1 if we approach it from the right?

NOTE $a \neq \pm\infty$!

Def One sided limits: Let $f: M \to \mathbb{R}$, $a \in \mathbb{R}$ be an accumulation point of M and $A \in \mathbb{R}^*$. We say that if $\forall \delta > 0 : P^+(a, \delta) \cap M \neq \emptyset$:

$$\lim_{x \to a^+} f(x) = A \iff \forall \varepsilon > 0 \ \exists \delta > 0 : f(P^+(a, \delta) \cap M) \subseteq U(A, \varepsilon)$$

" f has a one-sided limit A as x approaches a from the right " = right-handed limit

Symetrically, if $P^-(a,\delta)\cap M \neq \emptyset$ for every $\delta>0$,

$$\lim_{x\to a^-} f(x) = A \iff \forall \varepsilon>0\ \exists\delta>0 : f(P^-(a,\delta)) \subseteq U(A,\varepsilon).$$

„one-sided limit as x approaches a from the left"
= left-handed limit

Example: $\lim_{x\to 0^-} \operatorname{sgn} x = -1$ $\quad \lim_{x\to 0^+} \operatorname{sgn} x = 1$, $\lim_{x\to 0^+} \frac{1}{x} = \infty$, $\lim_{x\to 0^-}\frac{1}{x} = -\infty$

Remarks: If $\lim_{x\to a^+} f(x) = A$ and $\lim_{x\to a^-} f(x) = A$, then $\lim_{x\to a} f(x) = A$.

However, the converse does not hold: one of the one-sided limit might not be defined, e.g. because $P^-(a,\delta)\cap M = \emptyset$ for some $\delta>0$.

Nevertheless, if one-sided limits exist, they are equal to two sided. In particular, if one sided limits differ, two sided limit does not exist (as in case of $\operatorname{sgn} x$ in 0)

$\boxed{\text{Def}}$ (continuity at a point) Let $f: M\to\mathbb{R}$, $a\in M$.
We say that $f(x)$ is continuous at a point a, if

$$\forall\varepsilon>0\ \exists\delta>0 : f(U(a,\delta)\cap M) \subseteq U(f(a),\varepsilon).$$

In particular, if a is an accumulation point of M, f is continuous at a, if $\lim_{x\to a} f(x) = f(a)$.

{one-sided continuity:} f is left-continuous at a, if

$$\forall\varepsilon>0\ \exists\delta>0 : f(U^-(a,\delta)) \subseteq U(f(a),\varepsilon)$$

similarly, right-continuous with $+$

Theorem (uniqueness of a limit)
If $\lim_{x\to a} f(x)$ is defined, it is unique.

$\boxed{\text{Proof:}}$ Assume $f: M\to\mathbb{R}$, $a\in\mathbb{R}^*$ accumulation point of M and $A,B\in\mathbb{R}^*$ are such that $\lim_{x\to a} f(x) = A$ and $\lim_{x\to a} f(x) = B$.
Take $\varepsilon>0$ such that $U(A,\varepsilon)$ and $U(B,\varepsilon)$ are disjoint.
By definition of the limit, $\exists\delta>0$: $f(P(a,\delta)) \subseteq U(A,\varepsilon)\cap U(B,\varepsilon) = \emptyset$ ↯
since $f(P(a,\delta)) \neq \emptyset$ ∎

Connection between limits of sequences and functions

Theorem (not the poet Heine)

(Heine) Let $a \in \mathbb{R}^*$ be an accumulation point of M, $A \in \mathbb{R}^*$ and $f: M \to \mathbb{R}$. Then, the following two statements are equivalent:

(i) $\lim\limits_{x \to a} f(x) = A$

(ii) for every sequence $(x_n) \subseteq M$, such that $\lim\limits_{n\to\infty} x_n = a$, but $x_n \neq a \ \forall n$, it holds that $\lim\limits_{n\to\infty} f(x_n) = A$.

Applications:

- calculating limits of segences:

  $\lim\limits_{n\to\infty} n \cdot (e^{\frac{1}{n}} - 1) = 1$ by Heine theorem, since $\lim\limits_{n\to\infty} \frac{1}{n} = 0$

  and $\lim\limits_{x\to 0} \dfrac{e^x - 1}{x} = 1$.

- showing that a limit of a function does not exist:

  $\lim\limits_{x\to 0} \cos\frac{1}{x}$ does not exist. We find sequences $(a_n), (b_n)$ with $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = 0$, such that $\lim\limits_{n\to\infty} f(a_n) \neq \lim\limits_{n\to\infty} f(b_n)$.

  E.g., $a_n = \dfrac{1}{2\pi n}$, $b_n = \dfrac{1}{\pi(2n+1)}$.

  $\lim\limits_{n\to\infty} f(a_n) = 1$, $\lim\limits_{n\to\infty} f(b_n) = -1$.

$\boxed{\text{Proof:}}$

"$\Rightarrow$" Let $(x_n) \subseteq M$ be such that $\lim\limits_{n\to\infty} x_n = a$, $x_n \neq a \ \forall n$.

Given $\varepsilon > 0$, by (i) $\exists \delta$: $f(P(a,\delta) \cap M) \subseteq U(A, \varepsilon)$.

By assumption $\lim\limits_{n\to\infty} x_n = a$ and $(x_n) \subseteq M$, $\exists n_0 \ \forall n \geq n_0 : x_n \in P(a,\delta) \cap M$.

Thus, $\forall n \geq n_0$ $f(x_n) \in U(A, \varepsilon)$, i.e. $\lim\limits_{n\to\infty} f(x_n) = A$.

($\overset{A\in\mathbb{R}}{\Leftrightarrow} |f(x_n) - A| < \varepsilon$)

"$\Leftarrow$" By contrapositive: Assume (i) does not hold, i.e., $\exists \varepsilon > 0 \ \forall \delta \ \exists x \in P(a,\delta) \cap M$ s.t. $f(x) \notin U(A, \varepsilon)$.

For $n = 1, 2, \dots$ consider $\delta = 1/n$ and take x_n to be corresponding x.

Then $x_n \neq a \ \forall n$ and $\lim\limits_{n\to\infty} x_n = a$, but $|f(x_n) - A| > \varepsilon \ \forall n$, so $\lim\limits_{n\to\infty} f(x_n) \neq A$ ■

Theorem (arithmetics of limits of functions)

Let $a \in \mathbb{R}^*$, f, g are functions defined on some reduced neighborhood of a, $\lim_{x \to a} f(x) = A$ and $\lim_{x \to a} g(x) = B$, $A, B \in \mathbb{R}^*$. Then

(i) $\lim_{x \to a} (f(x) + g(x)) = A + B$, if it is defined

(ii) $\lim_{x \to a} (f(x) \cdot g(x)) = A \cdot B$, if it is defined

(iii) if $g(x) \neq 0$ on some reduced neighborhood of a,

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{A}{B}, \text{ if it is defined.}$$

Proof: combining Heine theorem and arithmetic of limits of sequences.

Theorem (limit of monotone function)

Let $a, b \in \mathbb{R}^*$, $a < b$, $f : (a,b) \to \mathbb{R}$ is monotone.

Then $\lim_{x \to a} f(x)$ and $\lim_{x \to b} f(x)$ exist.

[in particular, $\lim_{x \to a} f(a) = \inf(f((a,b)))$ and $\lim_{x \to a} f(b) = \sup(f((a,b)))$ if f is non-decreasing]

Proof similar to that for sequences.

Theorem (limit of a function and ordering)

Let f, g and h be functions defined on some reduced neighborhood of $a \in \mathbb{R}^*$.

(i) If $\lim_{x \to a} f(x) > \lim_{x \to a} g(x)$, then there exists $\delta > 0$, such that $f(x) > g(x)$ for $x \in P(a, \delta)$.

(ii) If $f(x) \geq g(x)$ for every $x \in P(a, \delta)$ (for some $\delta > 0$), then $\lim_{x \to a} f(x) \geq \lim_{x \to a} g(x)$, <u>if both limits exist</u>

(iii) (<u>sandwich</u>) If $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = A$ and $f(x) \leq g(x) \leq h(x)$ for every $x \in P(a, \delta)$ for some $\delta > 0$, $\lim_{x \to a} g(x) = A$, must exist.