Theorem (Leibniz criterion)

Let (a_n) is a non-increasing sequence with $\lim_{n\to\infty} a_n = 0$. Then $\sum (-1)^n a_n$ converges.

Consequence $\sum \left(-\frac{1}{n}\right)^n$ converges, but not absolutely

Rearranging series

- for finite sums, it is true that we can permute terms in any way and the ~~result will~~ sum will stay the same - by commutativity

NOTE Is the same true for series?

Def **Rearrangement** of a series $\sum_{n=1}^{\infty} a_n$ is a series $\sum_{n=1}^{\infty} a_{p(n)}$, where p is a permutation on $\mathbb{N}$ (i.e. a bijection $p: \mathbb{N} \to \mathbb{N}$).

Theorem (Riemann rearrangement theorem)
Let $\sum_{n=1}^{\infty} a_n$ be a series which converges, but does not converge absolutely. Then for every $\alpha \in \mathbb{R}^*$ there is a permutation $p: \mathbb{N} \to \mathbb{N}$, such that $\sum_{n=1}^{\infty} a_{p(n)} = \alpha$.
(Also $\sum a_{p(n)}$ might not have a sum at all.)

Observation: If $\sum a_n$ converges, but not absolutely, (a_n) contains infinitely many positive and infinitely many negative terms. Moreover, sum of the series formed by positive terms is $+\infty$, sum of the series formed by negative terms is $-\infty$.

Idea of proof of Riem. t.: By previous observation, we have infinite supply of positive and negative terms. We build rearranged series term by term, if the n-th partial sum is $> \alpha$, we add negative terms, if it is $< \alpha$, we add positive terms.

Theorem: (rearrangement of absolutely convergent series)
Let $\sum a_n$ be absolutely convergent. Then for every permutation p, $\sum a_{p(n)}$ is also absolutely convergent and has the same sum as $\sum a_n$.

Exponential function

- consider series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ for some $x \in \mathbb{R}$ (absolutely)
  - ratio test (for $x \neq 0$): $\lim_{n\to\infty} \frac{\left|\frac{x^{n+1}}{(n+1)!}\right|}{\left|\frac{x^n}{n!}\right|} = \lim_{n\to\infty} \frac{|x|}{n+1} = 0$
- series converges for every $x \in \mathbb{R}$, so we can define

$$\boxed{e^x = \exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}}$$

0^0 is not defined in general, here we define it as 1

Note • e is Euler's number $\left(\doteq 2{,}72,\ \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\right)$

Theorem (properties of exp)

(i) $\forall x, y \in \mathbb{R} \quad \exp(x+y) = \exp(x)\cdot\exp(y)$

(ii) $\exp(0) = 1$

(iii) $\forall x \quad \exp(-x) = \frac{1}{\exp(x)}$, and $\exp(x) > 0$

(iv) $\exp(x)$ is increasing and continuous function

(v) $\lim_{n\to\infty} \exp(n) = \infty$, $\lim_{n\to\infty} \exp(-n) = 0$

(vi) $\exp(x) \geq x+1$ (Moreover e is the only number with this property)

Proof of (i):

$$\sum_{n=0}^{\infty} \frac{(x+y)^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} n! \cdot \frac{x^k}{k!} \cdot \frac{y^{n-k}}{(n-k)!} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{x^k}{k!} \cdot \frac{y}{(n-k)!}^{n-k}$$

$$= \sum_{n=0}^{\infty} \sum_{n=k+l} \frac{x^k}{k!} \cdot \frac{y^l}{l!} = \sum_{k=0}^{\infty} \frac{x^k}{k!} \cdot \sum_{l=0}^{\infty} \frac{y^l}{l!}$$

distributivity of abs. conv. series

Theorem (infinite distributivity)

Let $\sum a_n$ and $\sum b_k$ be <u>absolutely convergent</u> series.

Then $\sum_{(n,k)\in\mathbb{N}^2} a_n b_k$ absolutely converges and its sum is

$\left(\sum a_n\right)\cdot\left(\sum b_k\right)$. (Note: By rearrangement of abs. conv. series, ordering of terms of $\sum a_n b_k$ does not really matter, one option – „diagonally"

$$\begin{matrix} a_0b_0 & a_0b_1 & a_0b_2 \\ a_1b_0 & a_1b_1 & a_1b_2 \\ a_2b_0 & a_2b_1 & a_2b_2 \end{matrix}$$

$$\sum_{l=0}^{\infty} \sum_{n+k=l} a_n b_k$$

Properties (iii) and (v) follow easily from (i) - Exercise.

Theorem (logarithm)
For every positive $y \in \mathbb{R}$ has equation $\exp(x) = y$ the unique solution. We denote this solution $\ln y$ - natural logarithm of y.

Theorem (exponential as a limit)
For every $x \in \mathbb{R}$, $\lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n = e^x$.

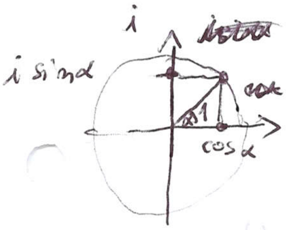
Geometric functions

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \cdots$$

Again, these definitions make sense for every $x \in \mathbb{R}$, since the series are absolutely convergent (by ratio test)

Interesting fact



exponential can be defined for complex numbers, so

$$e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = \sum_{\substack{n=0\\ n \text{ even}}}^{\infty} \frac{i^n x^n}{n!} + \sum_{\substack{n=0\\ n \text{ odd}}}^{\infty} \frac{i^n x^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$= \cos x + i \sin x.$$

This in particular, gives a relation between 5 most important numbers: $e^{i\pi} + 1 = 0$ ← complete!

Limit & continuity of a function

Recall: $f: M \to \mathbb{R}$
- $M \subseteq \mathbb{R}$ domain
- $\forall x \in M\ \exists!\, y \in \mathbb{R} : f(x) = y$.

- sequences were special functions $\mathbb{N} \to \mathbb{R}$.

Def Neighborhood of $x \in \mathbb{R}$ · in general, any open set containing x
- δ-neighborhood $U(x,\delta) = (x-\delta, x+\delta) = \{y \in \mathbb{R} \mid |x-y| < \delta\}$
- reduced δ-neighborhood $P(x,\delta) = (x-\delta, x) \cup (x, x+\delta)$

- <u>right</u> (reduced) δ-neighborhood $U^+(x,\delta) = [x, x+\delta)$
  $\left(P^+(x,\delta) = (x, x+\delta) \right)$
- symetrically, we define
  <u>left</u> δ-neighorhood $U^-(x,\delta) = (x-\delta, x]$
  $P^-(x,\delta) = (x-\delta, x)$

Neighorhood of infinities: $U(\infty,\delta) = (1/\delta, \infty)$
$U(-\infty,\delta) = (-\infty, -1/\delta)$

(reduced, left, right are the same / don't make sense)

<u>Remark</u> • ~~when~~ when δ gets smaller, ~~the~~ δ-neighborhoods get smaller

<u>Def</u> Let $M \subseteq \mathbb{R}$ be a non-empty set. We say that $a \in \mathbb{R}^*$ is <u>an accumulation (a limit) point</u> of M, if ~~∀~~ for every $\delta > 0$ we have $P(a,\delta) \cap M \neq \emptyset$.

[Equivalently, there exists a sequence (a_n), $a_n \in M$ $\forall n$, $a_n \neq a$, such that $\lim\limits_{n\to\infty} a_n = a$]

NOTE: • a does not have to be element of M • 1 is a lim. point of $(0,1)$
• some elements of M are not limit points:
set $\{1,2,3\}$ has no accumulation points

<u>Def.</u> Let $f: M \to \mathbb{R}$ and $a \in \mathbb{R}^*$ be an accumulation point of M and $A \in \mathbb{R}^*$.
We say that A is a limit of f ~~in~~ as x approaches a, ~~iff~~ we write $\lim\limits_{x \to a} f(x) = A$, if

$$\forall \varepsilon > 0 \ \exists \delta > 0 : f(\underbrace{P(a,\delta) \cap M}_{M \setminus \{a\}}) \subseteq U(A,\varepsilon).$$

In other words, $\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x \in Y: |x-a| < \delta \Rightarrow |f(x) - A| < \varepsilon$.