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Theorem (Linear combination of series)

Let $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ be series, and $\alpha, \beta \in \mathbb{R}$.

If $\alpha \neq 0$, then $\sum_{n=1}^{\infty} \alpha a_n$ converges if and only if $\sum_{n=1}^{\infty} a_n$ converges and $\sum_{n=1}^{\infty} \alpha a_n = \alpha \cdot \sum_{n=1}^{\infty} a_n$.

If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge, then $\sum_{n=1}^{\infty}(\alpha a_n + \beta b_n)$ converges, and $\sum_{n=1}^{\infty}(\alpha a_n + \beta b_n) = \alpha \sum_{n=1}^{\infty} a_n + \beta \sum_{n=1}^{\infty} b_n$.

$\boxed{\text{so } \sum \frac{1}{n^2} \text{ conv.} \Leftrightarrow \sum \frac{42}{n^2} \text{ konv.}}$

Proof: Exercise, use definition and limit arithmetic.

Question: Why we required $\alpha \neq 0$? (first part is $\Leftrightarrow$, second only $\Rightarrow$)!

Comparing ~~seque~~ series

- „analogue of sandwich theorem for limits"

Theorem (comparison criterion)

Let (a_n) and (b_n) be non-negative sequences.

(i) If $\exists n_0$ such that for every $n > n_0$: $a_n \leq b_n$ and $\sum b_n$ converges, then $\sum a_n$ also converges.

(ii) Let $\lim_{n\to\infty} \frac{a_n}{b_n} = l$. Then

a) • if $0 < l < +\infty$: ~~then~~ $\sum a_n$ converges, ~~then~~ if and only if $\sum b_n$ converges

b) • if $l = 0$: if $\sum b_n$ converges, then $\sum a_n$ converges

c) • if $l = +\infty$: if $\sum a_n$ converges, then $\sum b_n$ converges

Application: Recall: $\sum \frac{1}{n}$ diverges, $\sum \frac{1}{n^2}$ converges

$\sum c^n$ converges $\Leftrightarrow$ $|c| < 1$

Example 1: $a_n = \frac{1}{n^3}$ ~~[illegible]~~

take $b_n = \frac{1}{n^2}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{n^3}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{n^3} = 0$

so, by b) since $\sum \frac{1}{n^2}$ converges, $\sum \frac{1}{n^3}$ converges

Example 2 $a_n = \frac{1}{\sqrt{n}}$, take $b_n = \frac{1}{n}$, $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \sqrt{n} = \infty$,

so by c) contrapositive [$\sum b_n$ diverges $\Rightarrow \sum a_n$ div.]

we have that $\sum \frac{1}{\sqrt{n}}$ diverges since $\sum \frac{1}{n}$ diverges

Example 3 $a_n = \frac{1}{2n^2+n}$, take $b_n = \frac{1}{n^2}$, $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2}{2n^2+n} = \frac{1}{2}$

so by a) $\sum \frac{1}{n^2+n}$ converges

Conclusion: $\sum \frac{1}{n^\alpha}$ $\begin{cases} \text{converges} & \alpha \geq 2 \\ \text{diverges} & \alpha \leq 1 \end{cases}$ $\boxed{\text{actually } \alpha > 1}$ will see in summer

general approach for rational exp. a_n: find $b_n = n^\alpha$ s.t. $\frac{a_n}{b_n} \in (0,\infty)$

Example 4 What if limit does not exist:

$$\sum \frac{1+(-1)^n}{n^2} = 0 + \frac{2}{4} + 0 + \frac{2}{16} + 0 \dots$$

- $\frac{1+(-1)^n}{n^2} \leq \frac{2}{n^2}$ for every $n \in \mathbb{N}$
- $\sum \frac{2}{n^2}$ converges if and only if $\sum \frac{1}{n^2}$ converges by theorem about lin. comb., so $\sum \frac{1+(-1)^n}{n^2}$ conv.

Proof (i) define $s_k = \sum_{n=1}^{k} a_n$, $t_k = \sum_{n=1}^{k} b_k$. from $a_n \leq b_n$ for $n \geq n_0$, it follows that $s_k - s_{n_0} \leq t_k - t_{n_0}$.
If $\lim_{k\to\infty} t_k = c \in \mathbb{R}$, $t_k \leq c$ for every k, since (t_k) is monotone non-decreasing ~~increasing~~. Thus $s_k \leq c - t_{n_0} + s_{n_0}$, so (s_k) is bounded.
Since (s_k) is also non-decreasing, it is convergent.

(ii) follows from (i), since 1) $\exists n_0 \forall n \geq n_0$: $\frac{l}{2} < \frac{a_n}{b_n} < 2l$

2) $\frac{a_n}{b_n} < 1$

3) $1 < \frac{a_n}{b_n}$

Theorem (d'Alembert criterion / ratio test)

Let $a_n > 0\ \forall n$. Then

(i) If $\exists q \in (0,1)$ and $\exists n_0$ s.t. $\forall n > n_0$:

$$\frac{a_{n+1}}{a_n} < q, \text{ then } \sum a_n \text{ converges}$$

(ii) If $\limsup_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$, then $\sum a_n$ converges

(iii) If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$, then $\sum a_n$ converges

(iv) If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} > 1$, then $\sum a_n$ diverges

~~If lim~~ Remarks:

- There is a sequence (b_n), $b_n > 0\ \forall n$ such that

$$\limsup_{n\to\infty} \frac{b_{n+1}}{b_n} > 1 \text{ and } \sum b_n \text{ converges.}$$

- Test is inconclusive for $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$

Application $a_n = \frac{1000^n}{n!}$ $\quad \lim_{n\to\infty} \frac{\frac{1000^{n+1}}{(n+1)!}}{\frac{1000^n}{n!}} = \lim_{n\to\infty} \frac{1000}{n+1} = 0,$

so by (ii) $\sum \frac{1000^n}{n!}$ converges.

Does not apply: $a_n = \frac{1}{n^2}$ $\quad \lim_{n\to\infty} \frac{n^2+1}{n^2} = 1$ conv.

$a_n = \frac{1}{n}$ $\quad \lim_{n\to\infty} \frac{n+1}{n} = 1$ div

message: convergent series $\sum a_n$ does not mean (a_n) is monotone!

- $b_n = \begin{cases} \left(\frac{1}{2}\right)^n & \text{for } n \text{ even} \\ 4\left(\frac{1}{2}\right)^n & \text{for } n \text{ odd} \end{cases}$ $\quad$ $\overset{1}{2}, \overset{2}{\frac{1}{4}}, \overset{3}{\frac{1}{2}}, \overset{4}{\frac{1}{16}}, \frac{1}{8}, \dots$

$\frac{1}{8} \quad 2 \quad \frac{1}{8} \quad 2$

comparison crit

$\sum b_n \leq 4 \sum \left(\frac{1}{2}\right)^n \Rightarrow$ converges $\quad \Rightarrow \limsup_{n\to\infty} \frac{b_{n+1}}{b_n} = 2$

Theorem (Cauchy criterion / root ~~rule~~ test)

Let $a_n \geq 0$ for $\forall n$. Then

(i) If $\exists q \in (0,1)$ and $n_0 \in \mathbb{N}$: $\forall n \geq n_0$: $\sqrt[n]{a_n} < q$, then $\sum a_n$ converges

(ii) If $\limsup\limits_{n \to \infty} \sqrt[n]{a^n} < 1$, then $\sum a_n$ converges.

(iii) If $\lim\limits_{n \to \infty} \sqrt[n]{a^n} < 1$, then $\sum a_n$ converges.

(iv) If $\limsup\limits_{n \to \infty} \sqrt[n]{a^n} > 1$, then $\sum a_n$ diverges.

(v) If $\lim\limits_{n \to \infty} \sqrt[n]{a_n} > 1$, then $\sum a_n$ diverges.

Remark on proofs In both theorems (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii)

Note: ~~all~~ most previous criterions considered only series of non-negative elements

Def Series $\sum a_n$ converges absolutely if $\sum |a_n|$ converges.

~~Possibilities~~ **Theorem** (absolute convergence $\Rightarrow$ convergence)

If $\sum a_n$ converges absolutely, then $\sum a_n$ converges.

Proof: $\forall m > n$

$$|a_{n+1} + \dots + a_m| \overset{\Delta\text{-inequality}}{\leq} |a_{n+1}| + \dots + |a_m| = \big|\, |a_{n+1}| + \dots + |a_m| \,\big|$$

So, if $\sum |a_n|$ satisfies Bolzano-Cauchy condition, then $\sum a_n$ satisfies it as well. ■

So, series with neg. terms can be

- absolutely convergent
- convergent, but not absolutely
- divergent