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Limes superior & limes inferior

Def We say $\alpha \in \mathbb{R}^*$ is an <u>accumulation point</u> of a sequence (a_n), if (a_n) has a subsequence with limit (proper or improper) α.
We denote the set of accumulation points by H.

<u>Remark</u>: By Bolzano-Weierstrass theorem, every sequence has an accumulation point, i.e. $H \neq \emptyset$.
(If the sequence is not bounded, it has a subsequence with limit $\pm\infty$.)

<u>Observation</u>: H has maximum or $\infty \in H$, similarly H has minimum or $-\infty \in H$.

Proof: Let $\alpha = \sup(H)$. Assume $\alpha \notin H$. Then, there exists an accumulation point $\alpha_n \in [\alpha - \frac{1}{n}, \alpha]$ (or $[n, +\infty)$ if $\alpha = +\infty$) for every n, by def. of supremum (not bounded from above), that is, there exist sequences $(a_{1,n}), (a_{2,n}), \ldots$ such that $\lim_{n\to\infty} a_{i,n} = \alpha_i$ for every i and $(a_{i,n})$ is a subsequence of (a_n).
Thus, we can pick subsequence (b_n) of (a_n) such that $b_n \in [\alpha - \frac{2}{n}, \alpha]$ (or $[n-1, \infty)$), taking ~~b_n from~~ b_i from $(a_{i,n})$.
Then $\lim_{n\to\infty} b_n = \alpha$. Analogously for min.

Def. $\liminf_{n\to\infty} a_n = \min(H)$ <u>limes inferior</u>

$\limsup_{n\to\infty} a_n = \max(H)$ <u>limes superior</u>

<u>Remark</u> limsup and liminf always exist (unlike limit)

<u>Example</u> (a_n) defined as $a_n = n^{1+(-1)^n} + \frac{1}{n}$

$\limsup_{n\to\infty} a_n = +\infty$ (even terms)
$\liminf_{n\to\infty} a_n = 1$ (odd terms)

Equivalent definition of liminf and lim sup:

Let (b_n) be a sequence defined as $b_n = \sup(\{a_n, a_{n+1}, \dots\})$ and (c_n) be a seq. defined as $c_n = \inf(\{a_n, a_{n+1}, \dots\})$

if $+\infty$ or $-\infty$ we exceptionally allow $b_n = \infty$ or $c_n = -\infty$

Note that (b_n) is non-increasing and (c_n) is non-decreasing. Thus, (b_n) and (c_n) have limits (proper or improper).

<u>Theorem</u> (about liminf and limsup) A sequence a_n has a limit if and only if $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$.
Then $\liminf_{n\to\infty} a_n = \lim_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$.

Moreover, $\liminf_{n\to\infty} a_n = \lim_{n\to\infty} c_n$ and $\limsup_{n\to\infty} a_n = b_n$.
(regardless of whether (a_n) has a limit.)

Proof: Not too difficult, but we skip it for time reasons.

Series

Def Let (a_n) be an infinite sequence of reals. Then we say that expression $\sum_{n=1}^{\infty} a_n$ $(a_1 + a_2 + a_3 + \dots)$ is a <u>series</u>.

We define the <u>n-th partial sum</u> of the series to be $s_n = \sum_{i=1}^{n} a_i$ and <u>the sum of the series</u> as $\alpha = \lim_{n\to\infty} s_n$, if it exists, we write $\sum_{n=1}^{\infty} a_n = \alpha$.

We say a series is <u>convergent</u> if (s_n) has a <u>proper</u> limit (i.e. if s_n is convergent), otherwise we say a series is <u>divergent</u>.

<u>Remark</u> Series does not have to start from $n=1$: $\sum_{n=0}^{\infty} c_n, \sum_{n=m}^{\infty} b_n, \dots$.
Sum of the series depends on the starting point, unlike limit of the sequence.
However, starting point has no influence on whether the series is convergent or divergent.

Why we need to be careful

$$\sum_{n=0}^{n} \frac{1}{2^n} = S \qquad / \cdot \frac{1}{2} \qquad \sum_{n=0}^{n} \frac{1}{2^n} = 1 + \sum_{n=1}^{n} \frac{1}{2^n}$$

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{S}{2} \qquad / \qquad \text{so} \quad S = 1 + \frac{S}{2}$$

$$\Leftrightarrow \quad \underline{S = 2}$$

- this is a sum of geometric series, you probably already know.

$$\sum_{n=0}^{\infty} 2^n = S \qquad / \cdot 2 \qquad \sum_{n=0}^{n} 2^n = 1 + \sum_{n=1} \frac{1}{2^n}$$

$$\sum_{n=1}^{\infty} 2^n = 2S \qquad / \qquad \text{so} \quad S = 1 + 2S$$

$$\Leftrightarrow \quad \underline{S = -1}$$

- nonsense
  - by definition $\lim_{n\to\infty} \sum_{i=0}^{n} 2^i = \infty$
  - generally: if (a_n) is a sequence of non-negative reals, $\sum_{n=0}^{\infty} a_n$ exists (since (s_n) is non-decreasing) and it is non-negative real or $+\infty$

Further examples

$$\sum_{n=1}^{\infty} (-1)^n = -1 + 1 - 1 + 1 \ldots$$

intuitively $(-1+1) + (-1+1) \ldots = 0$

or $-1 + (1-1) + (1-1) = -1$

by definition $s_1 = -1$, $s_2 = 0$, …

$(s_n) = (-1, 0, -1, 0, \ldots)$

so $\lim_{n\to\infty} s_n$ does not exist!

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots \qquad \underline{\text{harmonic series}} \qquad \text{-vote!}$$

- DIVERGES: $s_m = \sum_{i=1}^{m} \frac{1}{i} = 1 + \sum_{k=1}^{r} \underbrace{\sum_{i=2^{k-1}+1}^{2^k} \frac{1}{i}}_{> \frac{1}{2} \text{ for every } k} \geq 1 + \frac{r}{2} = 1 + \frac{\log_2 m}{2}$, $m = 2^r$

$$\sum_{i=2^{k-1}+1}^{2^k} \frac{1}{i} > \sum_{i=2^{k-1}+1}^{2^k} \frac{1}{2^k} = 2^{k-1} \cdot \frac{1}{2^k} = \frac{1}{2}$$

so s_n is increasing, not bounded from above

$$\Rightarrow \lim_{n\to\infty} s_n = \infty$$

By previous lecture

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (we have shown that sequence of partial sums converges)

- $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$ converges (later)

- sequence (a_n) satisfies $a_n = 0$ for every $n \geq n_0$
  $\Rightarrow$ converges

Theorem (necessary condition for convergence of a series)

Let $\sum_{n=1}^{\infty} a_n$ be a ~~real~~ series of reals.

If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

Remarks:
- So, if $\lim_{n\to\infty} a_n \neq 0$, $\sum_{n=1}^{\infty} a_n$ diverges, e.g. $\sum_{n=1}^{\infty} (-1)^n$
- not sufficient !!! $\lim_{n\to\infty} \frac{1}{n} = 0$ and $\sum \frac{1}{n}$ diverges

Proof: Let $s = \sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n$. Then $a_n = s_n - s_{n-1}$,

so $\lim_{n\to\infty} a_n \overset{AL}{=} \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = s - s = 0$ ∎

Theorem (Cauchy condition for series)

Let $\sum_{n=1}^{\infty} a_n$ be a series of reals. Then $\sum a_n$ converges

if and only if $\forall \varepsilon > 0 \ \exists n_0 \ \forall m, n > n_0, m > n$:

$$\left|\sum_{i=n+1}^{m} a_i\right| < \varepsilon$$

Proof: $\sum a_n$ converges $\overset{\text{def.}}{\Longleftrightarrow}$ (s_n) converges $\overset{\text{Cauchy}}{\Longleftrightarrow}$
$\forall \varepsilon \ \exists n_0 \ \forall n, m > n_0 \ |s_n - s_m| < \varepsilon$ ∎

$|s_m - s_n| = \left|\sum_{i=n+1}^{m} a_i\right|$ if $m > n$