13/11/18

Arithmetic of Limits

- arithmetic of limits from tutorials worked only for proper limit, now, we would like to extend it to improper limits, so that we can for instance apply limit arithmetic, for $(a_n), (b_n)$ such that $\lim_{n\to\infty} a_n = \infty$ and $\lim_{n\to\infty} b_n = \infty$ to say that $\lim_{n\to\infty} a_n + b_n = „\infty + \infty" = \infty$.

Def Extended real numbers $\mathbb{R}^* = \{+\infty, -\infty\} \cup \mathbb{R}$

$+, \cdot$ and $<$ are defined as follows:

$\forall a \in \mathbb{R}: -\infty < a < \infty$

$\forall a \in \mathbb{R}^*, a \neq -\infty: \quad a + \infty = \infty + a = \infty$

$\forall a \in \mathbb{R}^*, a \neq \infty: \quad a + (-\infty) = (-\infty) + a = -\infty$

$\forall a \in \mathbb{R}^*, a > 0: \quad a \cdot (\pm\infty) = (\pm\infty) \cdot a = \pm\infty$

$\forall a \in \mathbb{R}^*, a < 0: \quad a \cdot (\pm\infty) = (\pm\infty) \cdot a = \mp\infty$

$\forall a \in \mathbb{R}: \quad \dfrac{a}{\pm\infty} = 0$

Undefined: $\infty + (-\infty)$, $(-\infty) + \infty$; $0 \cdot (\pm\infty)$, $(\pm\infty) \cdot 0$; $\dfrac{\pm\infty}{\pm\infty}$; $\dfrac{a}{0}$ $(a \in \mathbb{R}^*)$

Note: There is no arithmetic for $(a_n^{b_n})$

~~Also noted that~~ Exercise: Think of examples of sequences, where arithmetic of limits ~~would given~~ leads to undefined expression and various limits they can have.

E.g. $\lim_{n\to\infty} \dfrac{n}{n+1} = 1 = „\dfrac{\infty}{\infty}"$, but $\lim_{n\to\infty} \dfrac{n+1}{n^2} = 0 = „\dfrac{\infty}{\infty}"$ ...

Think about $„\dfrac{0}{0}"$.

Theorem (Extended limit arithmetic)

Let $(a_n), (b_n)$ be sequences of reals, $\lim_{n\to\infty} a_n = a \in \mathbb{R}^*$ and $\lim_{n\to\infty} b_n = b \in \mathbb{R}^*$. Then

1) $\lim_{n\to\infty} a_n + b_n = a + b$, if it is defined
2) $\lim_{n\to\infty} a_n \cdot b_n = a \cdot b$, if it is defined
3) $\lim_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{a}{b}$, if $b_n \neq 0 \ \forall n$ and $\dfrac{a}{b}$ is defined

Theorem (monotone subsequence)
Every sequence has a monotone subsequence.

- why is it interesting: ~~cm~~ recall that every monotone sequence has a limit, so

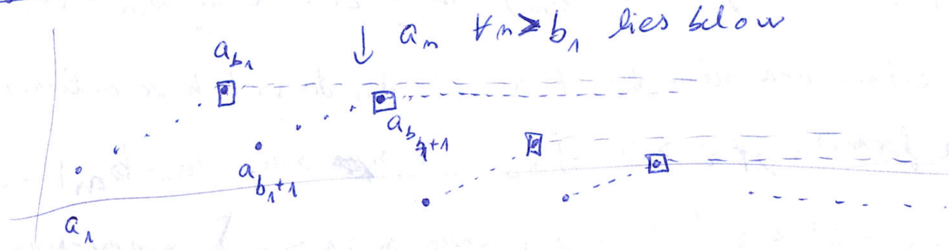
Theorem (Bolzano - Weierstrass) Every bounded sequence has a ~~convergenta~~ convergent subsequence.

Proof: Let (a_n) be any bounded sequence. By the previous theorem, it has a monotone subsequence (b_n), moreover (b_n) is bounded. So, by theorem about limit of a monotone sequence, (b_n) is convergent ∎

Proof (of mon. subsequence theorem) Let (a_n) be any sequence of reals. We will say that $k \in \mathbb{N}$ is „good", if there is an infinite non-decreasing subsequence of (a_n) starting with a_k.
That is, there exist $k = k_1 < k_2 < k_3 \dots$ such that $a_{k_1} \le a_{k_2} \le a_{k_3} \dots$,
We say $k \in \mathbb{N}$ is „bad" if there is a finite non-decreasing sequence starting with a_k, which cannot be extended, i.e., there are $k = k_1 < k_2 < k_3 < \dots k_m$, such that
$a_{k_1} \le a_{k_2} \le \dots \le a_{k_m}$ and $a_{k_m} > a_n$ for every $n > k_m$.
Observe, that every k is „good" or „bad" (can be both): we start a non-decreasing sequence from a_k and we either get stuck ($\Rightarrow$ „bad") or not ($\Rightarrow$ „good").
If ~~there~~ there is some „good" k, there is an infinite monotone (non-decreasing) subsequence. ✓
So, every k is „bad". Let b_k be the index of the last term of ~~the~~ a ~~non~~ non-decreasing sequence ~~from~~ starting with a_k, which cannot be extended: $a_{k_1} \le a_{k_2} \le \dots \le a_{b_k}$, $\forall n \ge b_k$ $a_n < a_{b_k}$
Consider sequence formed by $a_{b_1}, a_{b_{(b_1+1)}}, a_{b_{(b_{(b_1+1)}+1)}} \dots$
i.e., we start from a_{b_1}, then take last ~~element~~ term of sequence starting from ~~a_{b_1}~~ the next element a_{b_1+1}, a seq. last element of starting after this term ...

Note that by definition, these last indices form a decreasing sequence. ■

Figure



Def A sequence (a_n) of reals is Cauchy [COH shee] (or has the Cauchy property) if

$$\forall \varepsilon > 0 \ \exists n_0 : \forall n, m > n_0 : |a_m - a_n| < \varepsilon$$

I.e., terms of the sequence are getting arbitrarily close to each other.

Theorem (Cauchy ~~condition~~ property) A sequence (a_n) of reals is Cauchy if and only if it is convergent.

- A way how to show a limit exists without finding it. (conside sim

Example: Let $a_n = \sum_{k=1}^{n} \frac{1}{k^2}$ $\left(1, 1+\frac{1}{4}, 1+\frac{1}{4}+\frac{1}{9}, \dots\right)$

(limit is $\frac{\pi^2}{6}$, which we would have never guessed)

WLOG assume $n > m > n_0$

$$|a_n - a_m| = \sum_{k=m+1}^{n} \frac{1}{k^2} \leq \sum_{k=m+1}^{n} \frac{1}{k(k-1)} = \sum_{k=m+1}^{n} \left(\frac{1}{k-1} - \frac{1}{k}\right) = \frac{1}{m} - \frac{1}{n} < \frac{1}{m} < \frac{1}{n_0}$$

Since we know that $\forall \varepsilon > 0 \ \exists n_0 : \frac{1}{n_0} < \varepsilon$,

$\forall \varepsilon > 0 \ \exists n_0 \ \forall n, m > n_0 : |a_n - a_m| < \varepsilon$.

$\Rightarrow$ sequence converges, because it has the Cauchy property.

Proof "$\Leftarrow$" If $\lim_{n \to \infty} a_n = a \in \mathbb{R}$, then $\forall \varepsilon > 0 \ \exists n_0 \ \forall n, m > n_0$

$|a_n - a| < \varepsilon$ and $|a_m - a| < \varepsilon$, so by Δ-inequality

$$|a_n - a_m| \leq |a_n - a| + |a_m - a| < 2\varepsilon$$

$\Rightarrow$ sequence (a_n) is Cauchy.

"$\Rightarrow$" (more tricky)

If (a_n) is Cauchy, it is bounded (exercise).

Thus, by Bolzano-Weierstrass theorem, (a_n) contains a convergent subsequence (b_n). ~~with~~ Let $\lim_{n \to \infty} b_n = a$.

Then, (a_n) either has also limit a, or a_n doesn't have a limit.

We show the former: $\forall \varepsilon > 0 \ \exists n_0 : \forall n > n_0 \ |a_n - b_n| < \varepsilon$

and $|b_n - a| < \varepsilon$ ($b_n = a_{n'}$ where $n' \geq n > n_0$),

(combining conv. of (b_n) and Cauch. of (a_n))

Then $\quad |a_n - a| \leq |a_n - b_n| + |b_n - a| < 2\varepsilon$

Δ-inequality $\quad$ since $|a_n - b_n| < \varepsilon$ by Cauch.

$|b_n - a| < \varepsilon$ by limit ∎

Remark Sequence with improper limit is never Cauchy.