Theorem (uniqueness of limit)

Every sequence has at most one (proper or improper) limit.

Proof: We will show that a sequence cannot have two proper limits. Prove that a sequence cannot have proper and improper limit or ~~and~~ two improper limits as as exercise!

By contradiction: Assume a sequence (a_n) has two proper limits L_1 ~~an~~, $L_2 \in \mathbb{R}$, $L_1 \neq L_2$. WLOG $L_1 < L_2$.
Consider $\varepsilon = (L_2 - L_1)/3$. Then, by definition, there exist n_{01} and n_{02} such that $\forall n \geq n_{01}$: $|a_n - L_1| < \varepsilon$ and $\forall n \geq n_{02}$ $|a_n - L_2| < \varepsilon$. Thus, for $\forall n \geq \max(n_{01}, n_{02})$
$a_n > L_2 - \varepsilon$ and $a_n < L_1 + \varepsilon$. This is not possible, since by choice of ε, $L_1 + \varepsilon < L_2 - \varepsilon$. ↯

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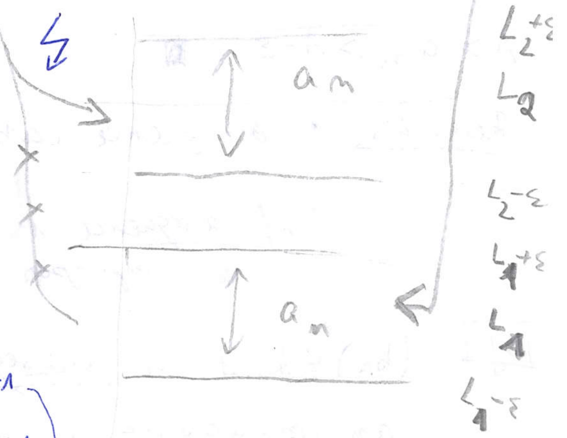

Properties of sequences

Def. A sequence (a_n) is

- non-decreasing if for every n: $a_n \leq a_{n+1}$
- increasing $a_n < a_{n+1}$
- non-increasing $a_n \geq a_{n+1}$
- decreasing $a_n > a_{n+1}$

equivalently: (by induction) $\forall n' > n$: $a_n \leq a_{n'}$

- monotone = non-increasing or non-decreasing
- bounded from above: $\exists K: \forall n$ $K \geq a_n$
  below: $\exists K$ $\forall n$ $K \leq a_n$
- bounded = bounded from above ~~or~~ and below
- eventually increasing / decreasing ...
  = $\exists n_0$ such that ~~for all~~ the sequence is increasing / decreasing ... from n_0 on
- e.g.: (a_n) defined ~~as~~ $a_n = n^2 - 4n$, i.e. $(-3, -4, -3, 0, 5, \dots)$, is not increasing, but is eventually increasing $(n_0 = 2)$

<u>Exercise</u>: Observe, that if a sequence is eventually bounded, it is bounded.

<u>Theorem</u> (limit of monotone sequence)
If $(a_n) \subseteq \mathbb{R}$ is eventually non-decreasing and bounded from above, then (a_n) converges.

<u>Note</u>: Symmetrically for (a_n) eventually non-~~in~~decreasing & bounded from below.

Proof: (a_n) is bounded, so, from completeness of $\mathbb{R}$, the set $\{a_n \mid n \in \mathbb{N}\}$ has supremum A.
By definition of supremum $\forall \varepsilon > 0 \ \exists n_0$ such that $A - \varepsilon < a_{n_0}$
(otherwise $A-\varepsilon$ is upper bound smaller than A)
Since (a_n) is non-decreasing, $\forall n \geq n_0' \ a_n \geq a_{n_0'}$, so $\forall n \geq \max(n_0, n_0')$:
$A > a_n > A - \varepsilon$. ∎

<u>Remarks</u>
- a sequence can be strictly increasing & bounded e.g. $1 - \frac{1}{n}$
- if a sequence is monotone & unbounded, it has improper limit (prove as exercise)

<u>Def</u> $(b_n) \subseteq \mathbb{R}$ is a <u>subsequence</u> of (a_n), if there exists an increasing function $f: \mathbb{N} \to \mathbb{N}$ such that $b_n = a_{f(n)}$.
In other words, there exists an <u>increasing</u> sequence of natural numbers $k_1, k_2 \dots$ such that $b_n = a_{k_n}$.

- For instance: $f(n) = 2n \to b_n = a_{2n}$ (every other element of (a_n))

<u>Remark</u>: "being a subsequence" is reflexive and transitive
is <u>not</u> antisymmetric (not even weakly)

<u>Exercise</u>: find two different sequences (a_n) and (b_n), such that (a_n) is a subsequence of (b_n) and (b_n) is a subsequence of (a_n).

Theorem (limit of a subsequence)

If (a_n) ~~is a~~ is a sequence ~~and~~ and (b_n) its subsequence

If ~~and~~ (a_n) has a limit (proper or improper), then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$.

Proof: Exercise [Note: slightly stronger statement than during tutorial.]

Application: If (a_n) has subsequences (b_n) and (c_n) with different limits, (a_n) does not have a limit.

Example $a_n = (-1)^n$ has subsequences (b_n), $b_n = 1 \; \forall n$

(c_n), $c_n = -1 \; \forall n$

$\to$ ~~[illegible]~~ (a_n) does not have a limit.

Computing limits

Theorem (arithmetic of proper limits)

Let $(a_n), (b_n) \subseteq \mathbb{R}$ be convergent sequences such that $\lim_{n\to\infty} a_n = a$, $\lim_{n\to\infty} b_n = b$. Then:

(i) sequence $(a_n + b_n)$ converges and $\lim_{n\to\infty} (a_n+b_n) = a+b$.

(ii) sequence $(a_n \cdot b_n)$ converges and $\lim_{n\to\infty} a_n \cdot b_n = a \cdot b$

(iii) if $b \neq 0$ and $b_n \neq 0 \; \forall n$, $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}$

[remark: if condition $b_n \neq 0$ holds only from some n_0 onwards, we can disregard the beginning of the sequence]

Proof (i) By triangle inequality,

$$|(a_n+b_n) - (a+b)| \leq |a_n - a| + |b_n - b|.$$

By assumptions, $\forall \varepsilon' > 0 \; \exists n_0 \; \forall n \geq n_0: |a_n - a| < \varepsilon'$ and $|b_n - b| < \varepsilon'$ [$\max(n_{0a}, n_{0b})$] Thus

$$\forall n \geq n_0 \quad |(a_n+b_n) - (a+b)| \leq |a_n - a| + |b_n - b| < 2\varepsilon'.$$

So, given ε, we consider $\varepsilon' = \frac{\varepsilon}{2}$ (so $2\varepsilon' = \varepsilon$), we find n_0 corresponding to ε' to get that

$$\forall n \geq n_0 \quad |(a_n+b_n) - (a+b)| < \varepsilon.$$

Note: We will use this trick with ε very often, next time, I will simply write ε instead ε' and we will not discuss multiple

(ii) $|a_n b_n - ab| = |(a_n - a)\cdot b_n + a(b_n - b)|$

$$\underset{\text{triangle ineq.}}{\leq} \underbrace{|a_n - a|}_{*}\,|b_n| + |a|\,\underbrace{|b_n - b|}_{\triangle}$$

by assumption,

$\forall \varepsilon > 0\ \exists n_0 : \forall n \geq n_0 \quad *$ and $\triangle < \varepsilon'$

Assume that $\varepsilon' < 1$ (if it holds for small ε', it holds for all).

so $\forall n \geq n_0 : \quad |b_n| < |b| + 1$

Thus $|a_n b_n - ab| < \varepsilon' \left(|b| + 1 + |a| \right)$

consider $\varepsilon' = \dfrac{\varepsilon}{|a| + |b| + 1}$.

(iii) $\left| \dfrac{a_n}{b_n} - \dfrac{a}{b} \right| = \dfrac{|(a_n - a)b + a(b - b_n)|}{|b_n|\,|b|} \leq \dfrac{|a_n - a|\,|b| + |a|\,|b - b_n|}{|b_n|\,|b|}$

- for given $0 < \varepsilon' < \dfrac{|b|}{2} \quad \exists n_0 : \forall n \geq n_0 : |a_n - a| < \varepsilon'$, $|b_n - b| < \varepsilon'$,

moreover $|b_n| > \dfrac{|b|}{2}$ $\left(|b_n| \quad |b|+\varepsilon,\ |b|,\ |b| - \varepsilon > \dfrac{|b|}{2} \right)$

Thus, $\left| \dfrac{a_n}{b_n} - \dfrac{a}{b} \right| < \dfrac{\varepsilon' (|a| + |b|)}{\frac{|b|^2}{2}}$ take $\varepsilon' = \dfrac{\varepsilon}{\frac{2(|a|+|b|)}{|b|^2}}$ ∎

Remarks: ! works only in one direction

$\lim_{n\to\infty} a_n + b_n = c \in \mathbb{R}$ does not mean $(a_n), (b_n)$ converge!

e.g. $a_n = (-1)^n,\ b_n = (-1)^{n+1}$

$a_n + b_n = 0$

$a_n = n \quad b_n = -n, \quad a_n + b_n = 0$

$a_n = n+1 \quad b_n = -n \quad a_n + b_n = 1$

Deducing limits from known limits

Theorem (limit and ordering)
Let $(a_n), (b_n)$ be two convergent sequences of reals with $\lim_{n\to\infty} a_n = a$, $\lim_{n\to\infty} b_n = b$. If there exists n_0 such that for every $n \geq n_0$: $a_n \leq b_n$, then $a \leq b$.

Remarks:
- „$\leq$" cannot be replaced by "$<$", in particular, there are sequences such that $\forall n \; a_n < b_n$, but $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$, for instance $a_n = \frac{1}{2n}$ and $b_n = \frac{1}{n}$. (both have limit 0)
- theorem holds even for improper limits: if $\lim_{n\to\infty} a_n = \infty$ and $\exists n_0 : \forall n \geq n_0 : a_n \leq b_n$, then $\lim_{n\to\infty} b_n = \infty$

Proof: By contradiction:
assume that $\exists n_0 : \forall n \geq n_0 : a_n \leq b_n$ and $b < a$.
Consider $\varepsilon = \frac{a-b}{3}$ (>0). Then $a - \varepsilon > b + \varepsilon$ and by definition of a limit, there is n_1 such that $\forall n \geq n_1$: $a_n > a - \varepsilon$ and $b_n < b + \varepsilon$.
It follows that for $n \geq \max(n_0, n_1)$ $a_n > b_n$, which contradicts our assumption $\lightning$ ∎

Exercise: Prove that if $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$, and $a < b$, then $\exists n_0 \; \forall n \geq n_0 : a_n < b_n$.

Theorem (Sandwich theorem / Two policemen theorem)
Let $(a_n), (b_n)$ and (c_n) be three real sequences satisfying: $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = a \in \mathbb{R}$ and $\exists n_0 \; \forall n \geq n_0$: $a_n \leq b_n \leq c_n$. Then (b_n) converges and $\lim_{n\to\infty} b_n = a$.

Proof: Observe that for an interval I, and $c, e \in I$, it holds that every d such that $c \le d \le e$ belongs to I.

By assumption of the theorem $\forall \varepsilon > 0\ \exists n_0\ \forall n \ge n_0 : a_n, c_n \in (a-\varepsilon, a+\varepsilon)$.

Thus, by previous observation $b_n \in (a-\varepsilon, a+\varepsilon)$.

It follows that $\lim_{n\to\infty} b_n = a$ ∎

Application: $\lim_{n\to\infty} \frac{1}{n!} = 0$: let $a_n = 0$, $b_n = \frac{1}{n!}$, $c_n = \frac{1}{n}$

Then $\forall n$: $a_n \le b_n \le c_n$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = 0$.

Thus, by the previous theorem, $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n!} = 0$.

Theorem (multiplying by limit zero)

Let (a_n) be bounded (not necessarily convergent) and (b_n) such that $\lim_{n\to\infty} b_n = 0$. Then $\lim_{n\to\infty} a_n b_n = 0$.

Exercise: prove the theorem.

Application: $\lim_{n\to\infty} \frac{\sin(n)}{n} = 0$ since $(\sin(n))$ is a sequence bounded by -1 and $+1$ and $\lim_{n\to\infty} \frac{1}{n} = 0$.