11. Lect. 8.1.19

Theorem (derivative & continuity) (*)

Let $a \in \mathbb{R}$, $\delta > 0$, $f : U(a,\delta) \to \mathbb{R}$ and there exists a <u>proper</u> (= finite) $f'(a) \in \mathbb{R}$. Then f is continuous at a.

[f is differentiable at a = has proper deriv.]

Mean value theorems

Let $a, b \in \mathbb{R}$, $a < b$, $f, g : [a,b] \to \mathbb{R}$ are continuous functions (on $[a,b]$) with derivative on (a,b).

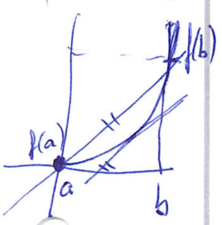
a) Rolle Theorem: If $f(a) = f(b)$, then there exists $c \in (a,b)$ s.t. $f'(c) = 0$.

b) Lagrange Theorem: There exists $c \in (a,b)$ s.t.

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

c) Cauchy Theorem: if g has proper non-zero derivative on (a,b), there exists $c \in (a,b)$ s.t.

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$



<u>Remarks</u>: Rolle Theorem is a special case of Lagrange th.

Lagrange theorem meaning: tangent at c is parallel to the straight line from $(a, f(a))$ to $(b, f(b))$

Proof: (Rolle) · If f is constant, the statement holds ($\forall c \in (a,b)$)

· assume f is not constant, i.e., there is x s.t. $f(a) \neq f(x)$.

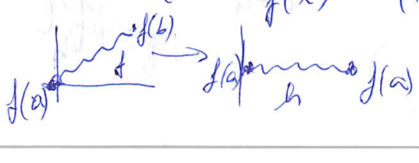
Assume $f(a) < f(x)$ (the other case analogous).

· ~~Hence~~ Since f is continuous on compact, it has global max. on $[a,b]$ at some point $c \in (a,b)$. By proposition from homework reading, if f has extreme in c and derivative at c, then $f'(c) = 0$. ∎

Lagrange: Apply Rolle theorem for $\cancel{F(x) = (f(x) - f(a))(b-a)}$

$$h(x) = f(x) - (x-a) \cdot \frac{f(b) - f(a)}{b - a}$$

∎



Theorem (l'Hospital Rule) • tool for finding limits "$\frac{0}{0}$" or "$\frac{\infty}{\infty}$"

Let $a \in \mathbb{R}^*$, f, g be functions defined on a reduced neigh. of a and have proper derivatives on it, moreover g' is nonzero on it.

Then: (i) if $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$ and $\lim_{x \to a} \frac{f'(x)}{g'(x)} = A \in \mathbb{R}^*$,

then $\lim_{x \to a} \frac{f(x)}{g(x)} = A$.

(ii) if $\lim_{x \to a} |g(x)| = \infty$ and $\lim_{x \to a} \frac{f'(x)}{g'(x)} = A \in \mathbb{R}^*$,

then $\lim_{x \to a} \frac{f(x)}{g(x)} = A$.

Remark • for instance $\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x}{1} = \underline{\underline{1}}$

• it cannot be applied if lim of ~~nom.~~ den. $\neq 0$ or ∞

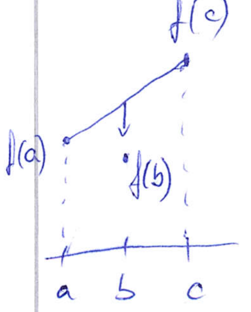
$\lim_{x \to 0} \frac{\sin x}{x+1} \neq \lim_{x \to 0} \frac{\cos x}{1}$!

• ~~it~~ can be ~~applied~~ useful for „$\infty - \infty$" and „$0 \cdot \infty$" limits

Higher order derivatives

• intuitively: we can treat derivative as a function, in particular, we can calculate derivative of derivative ...

$\boxed{\text{Def}}$ Let $a, \delta \in \mathbb{R}$, $\delta > 0$, $f: U(a, \delta) \to \mathbb{R}$ is a function. We define $f^{(0)} = f$ and for $n \in \mathbb{N}$ and $x \in U(a, \delta)$ we define $f^{(n)}(x) = \left(f^{(n-1)}(x)\right)'$, if $f^{(n-1)}$ is defined on some neighborhood of x. We call $f^{(n)}(x)$ n-th derivative of f at x.

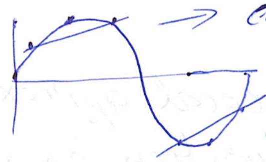


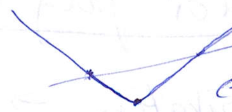
$\boxed{\text{Def}}$ Let $f: J \to \mathbb{R}$ be a function and J an interval. We say that f is convex (concave) on J if for every three points $a < b < c$ in J:

$$f(b) \leq f(a) + (f(c) - f(a)) \frac{b-a}{c-a}$$

$\geq$ ($\lneq$, $\gneq$ strictly)

Examples





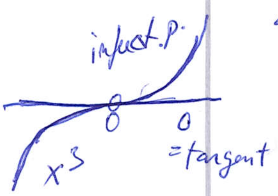
Theorem (convexity and 2nd derivative)

Let $-\infty \le a < b \le \infty$, $f:(a,b) \to \mathbb{R}$ s.t. f'' exists on (a,b) and f' is continuous on (a,b). Then

- $f'' \geq 0$ on (a,b) $\Rightarrow$ f convex (strictly if $>$) on (a,b)
  ≤ 0 on (a,b) $\Rightarrow$ f concave (strictly if $<$) on (a,b)

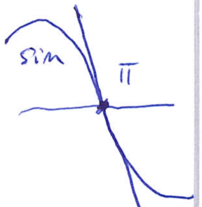
Note: f might be convex without having derivative (see ex. above)

$\boxed{\text{Def}}$ Inflection point: Let $a, \delta \in \mathbb{R}$, $\delta > 0$, $f: U(a,\delta) \to \mathbb{R}$.
We say that a is an inflection point of f, if $\exists$ proper $f'(a)$ and „a graph of f crosses the tangent at a"



i.e. $\exists \delta' \in (0,\delta]$ s.t. $x \in (a-\delta', a) \Rightarrow f(x) < f(a) + f'(a)(x-a)$
and $x \in (a, a+\delta') \Rightarrow f(x) > f(a) + f'(a)(x-a)$

or vice versa.



Theorem ($f'' \neq 0$ "$\overset{\text{necessary condition}}{\Rightarrow}$" not inflection) Let $a, \delta \in \mathbb{R}$, $\delta > 0$, $f: U(a,\delta) \to \mathbb{R}$ and $f''(a)$ exists but is not 0, then a is <u>not</u> an inflection point

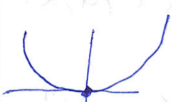
Theorem (sufficient condition for inflection)

If f has continuous derivative on (b,c), $a \in (b,c)$ and either $\forall x \in (b,a): f''(x) > 0$ and $\forall x \in (a,c): f''(x) < 0$

or $\forall x \in (b,a): f''(x) < 0$ and $\forall x \in (a,c)$ $f''(x) > 0$,

then a <u>is</u> an inflection point of f.

Remark: necessary condition is not sufficient.
it is possible that $f''(a) = 0$ and a is not an infl. point, e.g. x^4 has second deriv. 0 at 0, but it is not an infl. point.



Taylor polynomial

- derivative $\Rightarrow$ tangent = „best local approximation of a function by a line (= polyn. of deree 1)"
- best approx. by polynomial of degree k ?
  = Taylor polynomial

Definition: Let $a, \delta \in \mathbb{R}$, $\delta > 0$, $f: U(a,\delta) \to \mathbb{R}$, $n \in \mathbb{N}_0$ and there exists a proper n-th derivative $f^{(n)}(a) \in \mathbb{R}$ (for $n=0$ it means f continuous at a)

Taylor polynomial of degree n of f at a point a is
$$T_n^{f,a}(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \dots + \frac{f^{(n)}(a)(x-a)^n}{n!}$$

[note $T_n^{f,a}$ might have degree $< n$, if some derivatives are 0]

Theorem (characterisation of T. p.)

Let $a, \delta \in \mathbb{R}$, $\delta > 0$, $f: U(a,\delta) \to \mathbb{R}$, $n \in \mathbb{N}_0$ and $f^{(n)}(a) \in \mathbb{R}$ exist (and is proper). Then Taylor polynomial of degree n ~~the~~ $T_n^{f,a}(x)$ is the only polynomial of degree $\leq n$ satisfying
$$\lim_{x \to a} \frac{f(x) - P(x)}{(x-a)^n} = 0$$

[i.e. $f(x) - P(x) \in o((x-a)^n)$ as $x \to a$]

How precise exactly is the approximation?

Theorem (reminder of Taylor polynomial)

Let $n \in \mathbb{N}_0$, $a, \delta \in \mathbb{R}$, $\delta > 0$, $f, \varphi: U(a,\delta) \to \mathbb{R}$ with proper $f^{(n+1)}$, φ' on $U(a,\delta)$ and $\varphi' \neq 0$. Then $\forall x \in P(a,x)$ $\exists c \in (a,x)$ s.t.
$$R_n^{f,a}(x) = f(x) - T_n^{f,a}(x) = \frac{\varphi(x) - \varphi(a)}{n!\, \varphi'(c)} f^{(n+1)}(c)\,(x-c)^n.$$

~~Ex~~ useful choices of φ : $\varphi(t) = (x-t)^{n+1}$, $\varphi(t) = t$

Def Taylor series of a function ~~with~~ f at a point a

$$Tf_i^a(x) = \sum_{i=0}^{\infty} \frac{f^{(i)}(x)}{i!} (x-a)^i$$

(*) One sided derivatives $f'_+(a) = \lim_{x \to a^+} \frac{f(x) - f(a)}{x-a}$

f defined on $[a, a+\delta)$

on $(a-\delta, a]$ $f'_-(a) = \lim_{x \to a^-} \frac{f(x) - f(a)}{x-a}$

Theorem (limit of a derivative)

Let $a, \delta \in \mathbb{R}$, $\delta > 0$ $f : [a, a+\delta) \to \mathbb{R}$ is continuous from the right at a and f has proper derivative at $(a, a+\delta)$. ~~Then~~ and $\lim_{x \to a^+} f'(x) = A \in \mathbb{R}^*$. Then $f'_+(a) = A$.

E.g. we can calculate $f'_+(0)$ for $f(x) = \sqrt{x}$ (f not defined for $x < 0$, so $f'(0)$ does not exist).