Theorem (limit of composed function)

Let $a, A, B \in \mathbb{R}^*$, f be a function defined on a reduced neighborhood of A and $\lim_{x \to A} f(x) = B$, g be a function defined on a reduced neighborhood of a and $\lim_{x \to a} g(x) = A$.

If one of the following conditions hold:

1. $A, B \in \mathbb{R}$ and f is defined in A as $f(A) = B$ (i.e. f is contin. at A)
2. $\exists \delta > 0$ s.t. $g(x) \neq A$ for every $x \in P(a, \delta)$.

Then $\lim_{x \to a} f(g(x)) = B$.

Zoo of continuous functions

The following functions are continuous at every point of their domain:

- polynomials and their fractions
- e^x, $\ln x$, $\sin x$, $\cos x$

Operations preserving continuity

Theorem (operations preserving continuity)

Let $f, g : M \to \mathbb{R}$ be continuous at $a \in M$ and let $\alpha \in \mathbb{R}$.

Then $f+g$, $\alpha \cdot f$, $f \cdot g$, $\frac{f}{g}$ if $g(a) \neq 0$, $\max(f,g)$, $\min(f,g)$ and $|f|$ are functions continuous at a.

Explanation: $f+g\,(x) = f(x) + g(x)$

$\max(f,g)(x) = \max(f(x), g(x))$

$|f|(x) = |f(x)|$

Proofs: Follow from arithmetics of limits of functions or exercise.

Def: Let $f : M_1 \to \mathbb{R}$ and $g : M_2 \to \mathbb{R}$ be real functions such that $f(M_1) = \{ f(x) \mid x \in M_1 \} \subseteq M_2$. Then we can define a composition of f and g $\quad g \circ f : M_1 \to \mathbb{R}$ as $g \circ f(x) = g(f(x))$.

Theorem (composition preserves continuity)

Let $f : M_1 \to \mathbb{R}$ be continuous at a, $g : M_2 \to \mathbb{R}$ be continuous at $f(x) \in M_2$.

Then $g \circ f$ is continuous at ~~$g(f(x))$~~. a.

Def *Inverse function*: If f is an *injective* function $f: M \to \mathbb{R}$ (i.e. $\forall x, y \in M: x \neq y \Rightarrow f(x) \neq f(y)$), we define f^{-1} [not to be confused with $\frac{1}{f}$] as $f(x) = y \Leftrightarrow f^{-1}(y) = x$

Remark:
- $f^{-1}: f(M) \to M$
- if f is not injective, f^{-1} is not a function (several outputs for one value)
- if f is (strictly) increasing / decreasing, it is injective

Theorem (*continuity of inverse function*)
Let J be an interval, $f: J \to \mathbb{R}$ is increasing (decreasing) function continuous on J. Then $f^{-1}: f(J) \to \mathbb{R}$ is *continuous* and increasing (decreasing) on $f(J)$.

Question: What is $f(J)$? An interval

Theorem (*Darboux*) Let $a, b, y \in \mathbb{R}$, $a < b$, $f: [a,b] \to \mathbb{R}$ is a function continuous on $[a,b]$ such that $f(a) < y < f(b)$. Then there is $\alpha \in [a,b]$ such that $f(\alpha) = y$.

Proof: Let $\alpha := \sup(M)$, $M := \{x \in [a,b] \mid f(x) < y\}$.
- M bounded from above by $b \Rightarrow \sup(M)$ exists.
- $\alpha \neq a$, $\alpha \neq b$ since by continuity $\exists \delta > 0$ s.t.

  $\forall x \in [a, a+\delta): f(x) < y$ (consider $\varepsilon < y - f(a)$)

  $\forall x \in (b-\delta, b]: f(x) > y$ ($< f(b) - \varepsilon$)

So, $\alpha \in (a,b)$
- Assume $f(\alpha) \neq y$. By continuity in α:

  assume $f(\alpha) < y$, consider $\varepsilon < y - f(\alpha)$, then

  $\exists \delta > 0: x \in (\alpha - \delta, \alpha + \delta) \Rightarrow \underline{f(x) < f(\alpha) + \varepsilon < y}$

  ↯ definition of supremum ($\alpha + \frac{\delta}{2}$ should belong to M)

· assume $f(\alpha) > y$, consider $0 < \varepsilon < f(\alpha) - y$, then

$\exists \delta > 0 : x \in (\alpha - \delta, \alpha + \delta) \Rightarrow f(x) > f(\alpha) - \varepsilon > y$

$\lightning$ supremum (~~not true that $\forall \varepsilon>0\ \exists x \in M$ s.t. $\alpha-\varepsilon < x \le \alpha$~~ $(\alpha-\delta, \alpha+\delta) \cap M = \emptyset$

− either $\sup M \le \alpha - \delta$ or $\sup M > \alpha + \delta$) ■

Similarly if $f(a) > y > f(b)$.

Corollary (image of interval)

If I is an interval and $f: I \to \mathbb{R}$ is continuous on I, then $f(I) \subset \mathbb{R}$ is an interval

Proof: By Darboux theorem: if $u, v \in f(I)$, $u < v$, $\forall w$ s.t. $u < w < v$: $w \in f(I)$ ■

~~Theo~~ $\boxed{\text{Def}}$ We say that interval $[a,b]$ is compact if $a, b \in \mathbb{R}$, (= closed & bounded)

Motivation: later, we will extend this definition to compact sets.

Theorem (extremes on compact)

Let $a, b \in \mathbb{R}$, $a < b$ and $f: [a,b] \to \mathbb{R}$ is continuous on $[a,b]$. Then there exists $\alpha \in [a,b]$ such that $\forall x \in [a,b]$ if $f(x) \ge f(\alpha)$. Similarly, there exists $\beta \in [a,b]$ such that $\forall x \in [a,b]$ $f(x) \le f(\beta)$.

How to find α, β ? : homework

Derivatives

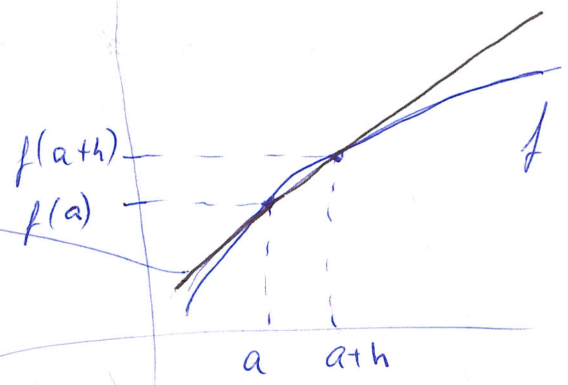
Def: Let $a \in \mathbb{R}$, $\delta > 0$ and $f: U(a, \delta) \to \mathbb{R}$. Derivative of f at a point a is a number $f'(a) := \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ if this limit exist.

$$= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Note: · One can define one sided derivatives $f'_+(a)$ and $f'_-(a)$ using one sided limits.

· $f'(a)$ can be $+\infty$ or $-\infty$

geometric interpretation



line passing through $(a, f(a))$ and $(a+h, f(a+h))$

$$y = f(a) + \boxed{\frac{f(a+h) - f(a)}{h}} \cdot h$$

(recall equation of line

$y = a + bx$

b ... ~~how speed~~ "steepness" of the slope)

Taking this to a limit $(h \to 0)$

$y = f(a) + f'(a)(x-a)$ is

a <u>tangent</u> of graph of the function f at a