Homework 8

Problem 2

Let $s_k = \sum_{n=1}^{k} a_n$ and $t_k = \sum_{n=1}^{k} b_n$. By assumption $\sum_{n=1}^{\infty} a_n = S$,

that is, $\lim_{k\to\infty} s_k = S$. Observe that $t_k = \sum_{n=1}^{k} b_n = \sum_{n=1}^{k} a_{3n-2} + a_{3n-1} + a_{3n}$

$= \sum_{n=1}^{3k} a_n = s_{3k}$. Thus, (t_k) is a subsequence of (s_k). By

theorem about limit of a subsequence $\lim_{k\to\infty} t_k = S$, so $\sum_{n=1}^{\infty} b_n = S$.

Problem 3

Assume $\lim_{n\to\infty} a_n = A \neq 0$, $A \in \mathbb{R}$. Then, $A = \frac{1}{2}\left(A + \frac{c}{A}\right)$ by arithmetic of limits.

Equivalently $A^2 = c$. Observing that $a_n > 0$ for every n, we conclude that if (a_n) has the proper limit, it is $\sqrt{c}$ or 0.

Experimentally, we can observe that for $n \geq 2$, (a_n) is non-increasing and greater ~~than~~ or equal to $\sqrt{c}$. We prove it:

i) By inequality between arithmetic and geometric mean, we have

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{c}{a_n}\right) \geq \sqrt{a_n \cdot \frac{c}{a_n}} = \sqrt{c}$$

, moreover, equality holds only for $a_n = \frac{c}{a_n}$, i.e. $a_n = \sqrt{c}$.

Thus, $a_n \geq \sqrt{c}$ for every $n \geq 2$, moreover, if $a_n \neq \sqrt{c}$, $a_{n+1} > \sqrt{c}$.

So, for $c \neq 1$, $a_n > \sqrt{c}$ for $\forall n \geq 2$, for $c = 1$, the sequence is constant.

ii) We show that if $a_n \geq \sqrt{c}$, $a_{n+1} \leq a_n$:

$$\frac{a_{n+1}}{a_n} = \frac{\frac{1}{2}\left(a_n + \frac{c}{a_n}\right)}{a_n} = \frac{1}{2} + \frac{1}{2}\cdot\frac{c}{a_n^2}.$$

If $a_n \geq \sqrt{c}$, $a_n^2 \geq c$, so $\frac{c}{a_n^2} \leq 1$.

Thus $\frac{a_{n+1}}{a_n} \leq 1 \iff a_{n+1} \leq a_n$.

We conclude that from $n = 2$, the sequence is non-increasing and bounded by $\sqrt{c}$ from below (so it cannot have limit 0). Thus it is convergent and limit is $\sqrt{c}$.

Note that for $n \geq 2$: $a_n \geq \sqrt{c} \geq \frac{c}{a_n}$.

Thus, if $a_n - \frac{c}{a_n} < 10^{-4}$, $a_n - \sqrt{c} < 10^{-4}$.