Problem 1:

a) $e^x \quad (e^x)' = (e^x)^{(2)} = \dots = e^x$

$e^0 = 1$

$$T^{e^x,0}(x) = \sum_{i=0}^{\infty} \frac{x^i}{i!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \dots = e^x$$

- converges for every $x \in \mathbb{R}$ (from winter)

b) $\frac{1}{1-x}$

$$\left(\frac{1}{1-x}\right)' = \left((1-x)^{-1}\right)' = (1-x)^{-2}$$

$$\left[(1-x)^{-2}\right]' = 2\,(1-x)^{-3}$$

$$\left[(1-x)^{-3}\right]' = 2\cdot 3\,(1-x)^{-4}$$

$$\left[(1-x)^{-1}\right]^{(k)} = k!\,(1-x)^{-(k+1)}$$

$x=0 \quad = k!$

$$T^{\frac{1}{1-x},0}(x) = \sum_{i=0}^{\infty} \frac{k!}{k!} x^i = 1 + x + x^2 + \dots$$

$$= \sum_{i=0}^{\infty} x^i$$

- converges $x \in (-1, 1)$, div. otherwise

c) $\sin 0 = 0$

		$x=0$
$(\sin x)' =$	$\cos x$	1
$(\cos x)' =$	$-\sin x$	0
$(-\sin x)' =$	$-\cos x$	-1
$(-\cos x)' =$	$\sin x$	0

conv. everywhere

$$0 + x + \frac{0}{2!}x^2 - \frac{1}{3!}x^3 + \frac{0}{4!}x^4 + \dots$$

$$T^{\sin x,0}(x) = \sum_{i=1}^{\infty} \frac{x^{2i-1}}{(2i-1)!}(-1)^{(i-1)}$$

d) $\ln(x+1) \overset{x=0}{=} 0$

$$[\ln(x+1)]' = \frac{1}{1+x}$$

$$\left[(x+1)^{-1}\right] = (-1)(x+1)^{-2}$$

$$\left[(-1)(x+1)^{-2}\right] = (-1)(-2)(x+1)^{-3}$$

$$\vdots$$

$$\left[\ln(x+1)\right]^{(k)} = (-1)^{(k+1)}(k-1)!\,(x+1)^{-k}$$

conv. $(-1, 1]$

$$T^{\ln(x+1),0} = \sum_{i=0}^{\infty} \frac{(i-1)!}{i!}(-1)^{i+1} x^i$$

$$= \sum \frac{(-1)^{i+1}}{i} x^i$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$$

e) Taylor series = polynomial itself

Problem 2 $T_3^{\sin,\pi} = -(x-\pi) + \frac{1}{6}(x-\pi)^3$

Problem 3

$$T_2^{f,0} = 1 - \frac{x}{2} - \frac{x^2}{8}$$

$f = \sqrt{1-x}$

calculator value $\doteq 0{,}9899494937$

a) $f' = \dfrac{-1}{2\sqrt{1-x}}$

$f'' = \dfrac{-1}{4(1-x)^{3/2}}$

$x = 0{,}02$

$1 - \frac{0{,}002}{2} = 0{,}99$

$1 - 0{,}01 - \frac{0{,}0004}{8} = \underline{0{,}9895}$:)

b) $T_3^{f,0}(x) = x - \frac{x^2}{2} + \frac{x^3}{3}$

calc $\ln 1{,}2 \doteq 0{,}18232$

$x = 0{,}2$

$x - \frac{x^2}{2} = 0{,}2 - \frac{0{,}04}{2} = 0{,}18$

$x - \frac{x^2}{2} + \frac{x^3}{3} = 0{,}182\overline{6}$

Problem 4

$$R_n^{f,a}(x) = \frac{\varphi(x) - \varphi(a)}{n!\,\varphi'(c)} f^{(n+1)}(c)\,(x-c)^n$$

take $\varphi(t) = (x-t)^{n+1}$ $\quad \varphi'(t) = -(x-t)^n \cdot (n+1)$

$c \in (a,x)$

[φ' vlastní nenulová]

Lagrange reminder

$$R_n^{f,a}(x) = \frac{-(x-a)^{n+1}}{n! \cdot (-(x-c)^n)(n+1)} f^{(n+1)}(c)\,(x-c)^n$$

$$= \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(c)$$

$$R_3^{\sin x,0} = \frac{x^4}{4!} \cdot \sin^{(4)}(c) < \frac{x^5}{4!}, \quad c \in (0,\ 0{,}1).$$

$\sin^{(4)} x = \sin x$

$\sin x < x$

$x = 0{,}1 \Rightarrow R_3^{\sin x,0} < \frac{0{,}00001}{4!}$:)

(or can even calculate for what x the estimate is precise enough)

See kam.mff.cuni.cz/~shirka for solution of Problem 5b