Lecture 7 (3.4.2019)

(partially translated and adapted from lecture notes by Martin Klazar)

Multivariable calculus

We will work in m-dimensional *Euclidean space* $\mathbb{R}^m$, $m \in \mathbb{N}$, which is a set of all ordered m-tuples of reals $\mathbf{x} = (x_1, x_2, \ldots, x_m)$ with $x_i \in \mathbb{R}$. It is an m-dimensional vector space over $\mathbb{R}$ — we can sum and subtract its elements and we can multiply them by real constants. We introduce a notion of distance in $\mathbb{R}^m$, using *(Euclidean) norm* wich is a mapping $\|\cdot\|: \mathbb{R}^m \to [0, +\infty)$ defined as

$$\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2 + \ldots + x_m^2}\ .$$

Euclidean norm has the following properties ($a \in \mathbb{R}$, $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$):

(i) (positivity) $\|\mathbf{x}\| \geq 0$ a $\|\mathbf{x}\| = 0 \iff \mathbf{x} = \mathbf{o} = (0, 0, \ldots, 0)$,

(ii) (homogenity) $\|a\mathbf{x}\| = |a| \cdot \|\mathbf{x}\|$ and

(iii) (triangle inequality) $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$.

Using the norm, we define *(Euclidean) distance* $d(\mathbf{x}, \mathbf{y}): \mathbb{R}^m \times \mathbb{R}^m \to [0, +\infty)$ between two points x and y in $\mathbb{R}^m$ as

$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \ldots + (x_m - y_m)^2}\ .$$

Properties of Euclidean distance ($\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^m$):

(i) (positivity) $d(\mathbf{x}, \mathbf{y}) \geq 0$ and $d(\mathbf{x}, \mathbf{y}) = 0 \iff \mathbf{x} = \mathbf{y}$,

(ii) (symmetry) $d(\mathbf{x}, \mathbf{y}) = d(\mathbf{y}, \mathbf{x})$ and

(iii) (triangle inequality) $d(\mathbf{x}, \mathbf{y}) \leq d(\mathbf{x}, \mathbf{z}) + d(\mathbf{z}, \mathbf{y})$.

With exception of triangle inequality (deriving of which requires more effort), these properties of norm and distance follow easily form the definition.

(Open) ball $B(\mathbf{a}, r)$ with radius $r > 0$ and center $\mathbf{a} \in \mathbb{R}^m$ is the set of points in $\mathbb{R}^m$ with distance from $\mathbf{a}$ less than r:

$$B(\mathbf{a}, r) = \{\mathbf{x} \in \mathbb{R}^m \mid \|\mathbf{x} - \mathbf{a}\| < r\}\ .$$

Open set in $\mathbb{R}^m$ is a subset $M \subset \mathbb{R}^m$ such that for every point $\mathbf{x} \in M$ there is a ball with center $\mathbf{x}$ contained in M:

$$M \text{ is open} \iff \forall \mathbf{x} \in M\ \exists r > 0: \ B(\mathbf{x}, r) \subset M\ .$$

Following properties of open sets in $\mathbb{R}^m$ can be derived as a simple exercise:

(i) sets $\emptyset$ a $\mathbb{R}^m$ are open,

(ii) union $\bigcup_{i\in I} A_i$ of any system $\{A_i \mid i \in I\}$ of open sets A_i is an open set

(iii) intersection of two (finitely many) open sets is an open set.

Intersection of infinitely many open sets might not be open. *Neighborhood of a point* $\mathbf{a} \in \mathbb{R}^m$ is any open set in $\mathbb{R}^m$ containing $\mathbf{a}$.

We will consider functions $f: M \to \mathbb{R}$, $f = f(x_1, x_2, \dots, x_m)$, defined on $M \subset \mathbb{R}^m$ and mappings

$$f: M \to \mathbb{R}^n,\ M \subset \mathbb{R}^m,\ f = (f_1, f_2, \dots, f_n)\ ,$$

where $f_i = f_i(x_1, x_2, \dots, x_m)$ are *coordinate functions*. Our goal will be to generalize derivative as a linear approximation and a notion of integral to functions of several variables.

First, we generalize concept of continuity. Let $U \subset \mathbb{R}^m$ be a neighborhood of a point $\mathbf{a} \in \mathbb{R}^m$. We say that a function $f: U \to \mathbb{R}$ is *continuous at* $\mathbf{a}$, if

$$\forall \varepsilon > 0\ \exists \delta > 0 :\ \|\mathbf{x} - \mathbf{a}\| < \delta \Rightarrow |f(\mathbf{x}) - f(\mathbf{a})| < \varepsilon\ .$$

More generally, a mapping $f: U \to \mathbb{R}^n$, is *continuous at* $\mathbf{a}$ if

$$\forall \varepsilon > 0\ \exists \delta > 0 :\ \|\mathbf{x} - \mathbf{a}\| < \delta \Rightarrow \|f(\mathbf{x}) - f(\mathbf{a})\| < \varepsilon\ ,$$

i.e., we replace absolute value (which is the norm in $\mathbb{R}^1$) by norm in $\mathbb{R}^n$.
Similarly, we can generalize the notion of limit of a function:

$$\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = c \Leftrightarrow \forall \varepsilon > 0\ \exists \delta > 0 : \mathbf{x} \in B(\mathbf{a}, \delta) \setminus \{\mathbf{a}\} \Rightarrow |f(\mathbf{x}) - c| < \varepsilon.$$

Multivarible Riemann Integral

First, we generalize a notion of Riemann integral to multivariable functions, defining multivariable analogues of partition of an interval and upper and lower Riemann sum.

An *n-dimensional box* is a Cartesian product of closed intervals

$$I = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$$

where $-\infty < a_i < b_i < \infty, i = 1, \dots, n$. For instance, for 2-dimensional box is a closed rectangle with sides parallel to the axes.

Volume of a box is defined as $|I| = \prod_{i=1}^{n}(b_i - a_i)$. A *partition* of a box is a set of boxes

$$D = \{[c_1^{j_1}, c_1^{j_1+1}] \times \cdots \times [c_n^{j_n}, c_n^{j_n+1}] | 0 \le j_i < k_i, 1 \le i \le n\},$$

where $a_i = c_i^0 < c_i^1 < \cdots < c_i^{k_i} = b_i$ are some partitions of the intervals $[a_i, b_i]$, $i = 1, \dots, n$. *Norm* of a partition is defined as

$$\lambda(D) = \max_{0\le j_i<k_i, 1\le i\le n} (c_i^{j+1} - c_i^j),$$

i.e., as a maximal "length of an edge of a sub-box".

One can now define a partition with points and generalize a Riemann definition of integral. However, we will proceed by generalizing Darboux definition of the integral.

Let I be a box with a partition D and let $f : I \to \mathbb{R}$ be a function. For every box $J \in D$ we define $m(J) = \inf_{\mathbf{x}\in J} f(\mathbf{x})$ and $M(J) = \sup_{\mathbf{x}\in J} f(\mathbf{x})$. We define lower and upper Riemann sum as

$$s(f, D) = \sum_{J\in D} |J| \cdot m(J), \quad S(f, D) = \sum_{J\in D} |J| \cdot M(J)$$

and lower and upper integral as

$$\underline{\int_I} f = \sup(\{s(f, D) | D \text{ is a partition of } I\}),$$

$$\overline{\int_I} f = \inf(\{S(f, D) | D \text{ is a partition of } I\}).$$

Similarly as in one dimension, the following inequalities hold

$$s(f, D) \leq \underline{\int_I} f \leq \overline{\int_I} f \leq S(f, D).$$

Integral of f on I is then defined as a real number

$$\int_I f = \underline{\int_I} f = \overline{\int_I} f$$

if upper integral equals lower integral.

We denote the set of functions which have integral on I by $\mathcal{R}(I)$.

We say that a set $E \subseteq \mathbb{R}^m$ has *measure zero* if for every $\varepsilon > 0$ exists a sequence of boxes $I_1, I_2, \dots$ in $\mathbb{R}^m$, such that $\sum_{n=1}^{\infty} |I_n| < \varepsilon$ and $E \subset \cup_{n=1}^{\infty} I_n$.

Theorem 28. *Let $I \subseteq \mathbb{R}^m$ be a box and $f : I \to \mathbb{R}$ is a well defined function. Then $f \in \mathcal{R}(I)$ if and only if f is bounded and a set its points of discontinuity has measure zero.*

Integral over a bounded set $E \subset \mathbb{R}^m$ which is not a box: A *characteristic function* of a set E is a function $\chi_E : \mathbb{R}^m \to \{0, 1\}$ defined as $\chi_e(\mathbf{x}) = 1$ if $\mathbf{x} \in E$ and $\chi_e(\mathbf{x}) = 0$ otherwise. Let I be a box containing E. *Volume* of E is defined as $vol(E) = \int_I \chi_E$, if the integral exists. Finally, we define $\int_E f = \int_I f(\mathbf{x}) \cdot \chi_E$.