Lecture 6 (27.3.2019)

(translated and slightly adapted from lecture notes by Martin Klazar)

Applications of integrals

We estimate *factorial* $n! = 1 \cdot 2 \cdot \ldots \cdot n$ as follows: for $f(x) = \log x : [1, +\infty) \to [0, +\infty)$ and a partition $D = (1, 2, \ldots, n+1)$ of interval $[1, n+1]$ we have

$$s(f, D) = \sum_{i=1}^{n} 1 \cdot \log i = \log(n!) \quad \text{a} \quad S(f, D) = \sum_{i=1}^{n} 1 \cdot \log(i+1) = \log((n+1)!) \ .$$

Since $s(f, D) < \int_1^{n+1} \log x = (n+1)\log(n+1) - (n+1) + 1 < S(f, D)$, for $n \geq 2$ we get estimate

$$n \log n - n + 1 < \log(n!) < (n+1)\log(n+1) - n$$

and so

$$e\left(\frac{n}{e}\right)^n < n! < e\left(\frac{n+1}{e}\right)^{n+1} \ .$$

Similarly we estimate *harmonic numbers* H_n,

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} \ .$$

For a function $f(x) = 1/x : (0, +\infty) \to (0, +\infty)$ and a partition $D = (1, 2, \ldots, n+1)$ of interval $[1, n+1]$ we have that

$$s(f, D) = \sum_{i=1}^{n} 1 \cdot \frac{1}{i+1} = H_{n+1} - 1 \quad \text{a} \quad S(f, D) = \sum_{i=1}^{n} 1 \cdot \frac{1}{i} = H_n \ .$$

Since $s(f, D) < \int_1^{n+1} 1/x = \log(n+1) < S(f, D)$, for $n \geq 2$ we get

$$\log(n+1) < H_n < 1 + \log n \ .$$

Similarly one can estimate also sums of infinite series, but we need integral over infinite domain to do that.

For $a \in \mathbb{R}$ and $f : [a, +\infty) \to \mathbb{R}$ such that $f \in \mathcal{R}(a, b)$ for every $b > a$, we define

$$\int_a^{+\infty} f := \lim_{b \to +\infty} \int_a^b f \ ,$$

if the limit exists (we allow $\pm\infty$). We say that the integral converges if and only if the limit is a real number and we say that the integral diverges otherwise.

Theorem 26 (Integral criterion of convergence)**.** *Let a be and integer and $f : [a, +\infty) \to \mathbb{R}$ be a function which is non-negative and non-increasing on $[a, +\infty)$. Then,*

$$\sum_{n=a}^{\infty} f(n) = f(a) + f(a+1) + f(a+2) + \ldots < +\infty \iff \int_a^{+\infty} f < +\infty \ .$$

So, the series converges if and only if the corresponding integral converges.

Proof. The sequence of partial sums of the series is non-decreasing and therefore it has a limit which is either real or $+\infty$. Since f is monotone, $f \in \mathcal{R}(a, b)$ for every real $b > a$. Moreover, since f is non-negative, $\int_a^{b'} f \geq \int_a^b f$, if $b' \geq b$. Then $\lim_{b\to+\infty} \int_a^b f$ exists and is either real or $+\infty$. For some integer $b > a$, consider the partition $D = (a, a+1, a+2, \ldots, b)$ of $[a, b]$. We have the following inequalities:

$$\sum_{i=a+1}^{b} f(i) = s(f, D) \leq \int_a^b f \leq S(f, D) = \sum_{i=a}^{b-1} f(i) \ .$$

It follows that bounded partial sums imply bounded integrals $\int_a^b f$ for any integer $b > a$ and the other way round. Thus, both limits are either real or $+\infty$. □

Now we can easily decide convergence of

$$\sum_{n=1}^{\infty} \frac{1}{n^s}, \ s > 0 \ .$$

For $s \neq 1$, we have

$$\int_1^{+\infty} \frac{dx}{x^s} = \frac{x^{1-s}}{1-s}\bigg|_1^{+\infty} = (1-s)^{-1}(\lim_{x\to+\infty} x^{1-s} - 1) \ ,$$

this equals $+\infty$ for $0 < s < 1$ and $(s-1)^{-1}$ for $s > 1$. For $s = 1$ we have

$$\int_1^{+\infty} \frac{dx}{x} = \log x|_1^{+\infty} = \lim_{x\to+\infty} \log x = +\infty \ .$$

Thus, by integral criterion the series converges if and only if $s > 1$.

Next, consider the series

$$\sum_{n=2}^{\infty} \frac{1}{n \log n} \ .$$

Here,

$$\int_2^{+\infty} \frac{dx}{x \log x} = \log\log x|_2^{+\infty} = \lim_{x\to+\infty} \log\log x - \log\log 2 = +\infty \ .$$

By integral criterion the series diverges. Exercise: analyze convergence of $\sum_{n\geq 2} 1/n(\log n)^s$, $s > 1$.

We have already shown estimates of factorial using integrals. Now we show how to extend factorial to a smooth function on $[1, +\infty)$.

Theorem 27 (Gamma function)**.** *Function* Γ *defined as*

$$\Gamma(x) := \int_0^{+\infty} t^{x-1}e^{-t}\, dt : \ [1, +\infty) \to (0, +\infty)$$

satisfies the following functional equation

$$\Gamma(x + 1) = x\Gamma(x) \ .$$

on interval $[1, +\infty)$*. Moreover,* $\Gamma(1) = 1$ *and* $\Gamma(n) = (n - 1)!$ *for integers* $n \geq 2$*.*

Proof. First, we show that $\Gamma(x)$ is correctly defined. For every fixed $x \geq 1$, the integrand is a non-negative continuous function (for $x = 1$ and $t = 0$ we let $0^0 = 1$). Since $\lim_{t\to+\infty} t^{x-1}e^{-t/2} = 0$ (exponential grows faster than a polynomial), for every $t \in [0, +\infty)$ we have the following inequality:

$$t^{x-1}e^{-t} = t^{x-1}e^{-t/2} \cdot e^{-t/2} \leq ce^{-t/2} \ ,$$

where $c > 0$ is a constant depending only on x. Thus, integrals over finite intervals $[0, b]$ are defined, for $b \to +\infty$ don't decrease and have a finite limit:

$$\int_0^b t^{x-1}e^{-t}\, dt \leq \int_0^b ce^{-t/2} = c(1 - e^{-b/2}/2)\, dt < c \ .$$

The value $\Gamma(x)$ is therefore defined for every $x \geq 1$. For $x = 1$, we have

$$\Gamma(1) = \int_0^{+\infty} e^{-t}\, dt = (-e^{-t})|_0^{+\infty} = 0 - (-1) = 1 \ .$$

Functional equation can be derived by integration per partes:

$$\begin{aligned}
\Gamma(x+1) &= \int_0^{+\infty} t^x e^{-t}\, dt = t^x(-e^{-t})|_0^{+\infty} - \int_0^{+\infty} xt^{x-1}(-e^{-t})\, dt \\
&= 0 - 0 + x\int_0^{+\infty} t^{x-1}e^{-t}\, dt \\
&= x\Gamma(x) \ .
\end{aligned}$$

Values $\Gamma(n)$ follow by induction. □

Note that extending factorial to a function f on $[1, +\infty)$ satisfying $f(x + 1) = xf(x)$ can be done in many ways, starting from any function defined on

$[1, 2)$ with $f(1) = 1$ and extending it. The advantage of $\Gamma(x)$ is that it has derivatives of all orders.

Finally, we give formulas for area, length of a curve and volume of solids of revolution. We have essentially defined area $U(a, b, f)$ (that is, points (x, y) in a plane satisfying $a \le x \le b$ a $0 \le y \le f(x)$) under the graph of function f as $\int_a^b f$.

For a function $f: [a, b] \to \mathbb{R}$ we define length of its graph $G = \{(x, f(x)) \in \mathbb{R}^2 \mid a \le x \le b\}$ as a limit of length of a sequence of broken lines L with endpoints of segments on G which "approximate G", where the length of a longest segment of L tends to 0. For "nice" functions f (for instance those with continuous derivative), this limit exists and we can calculate it using Riemann integral. A segment of L connecting points $(x, f(x))$ and $(x + \Delta, f(x + \Delta))$ has by Pythagoras theorem length

$$\sqrt{\Delta^2 + (f(x + \Delta) - f(x))^2} = \Delta \sqrt{1 + \left(\frac{f(x + \Delta) - f(x)}{\Delta}\right)^2} .$$

From this, one can derive the following formula:

Theorem (length of a curve). *Let $f: [a, b] \to \mathbb{R}$ be a function with continuous derivative on $[a, b]$ (so $\sqrt{1 + (f')^2} \in \mathcal{R}(a, b)$). Then*

$$\text{délka}(\{(x, f(x)) \in \mathbb{R}^2 \mid a \le x \le b\}) = \int_a^b \sqrt{1 + (f'(t))^2}\, dt .$$

For a subset $M \subset \mathbb{R}^3$ we can define its volume as a limit, for $n \to \infty$, of the sume of volumes of $1/n^3$ cubes K in the set

$$\{K = [\tfrac{a}{n}, \tfrac{a+1}{n}] \times [\tfrac{b}{n}, \tfrac{b+1}{n}] \times [\tfrac{c}{n}, \tfrac{c+1}{n}] \mid a, b, c \in \mathbb{Z} \;\&\; K \subset M\} .$$

If M is "nice", this limit exists and can be computed using integral. In particular, if M is obtained by rotating some planar figure around the horizontal axis, we get the following.

Theorem (volume of solid of revolution). *Let $f \in \mathcal{R}(a, b)$ and $f \ge 0$ on $[a, b]$. For a volume of a body defined as*

$$V = \{(x, y, z) \in \mathbb{R}^3 \mid a \le x \le b \;\&\; \sqrt{y^2 + z^2} \le f(x)\}$$

obtained by rotating a planar figure $U(a, b, f)$ under the graph of a function f around x-axis we have

$$\text{volume}(V) = \pi \int_a^b f(t)^2\, dt .$$

The formula can be obtained by cutting V by planes perpendicular to x-axis into slices of length $\Delta > 0$ and summing their volumes. Each slice is roughly a cylinder with radius $|f(x)|$ and height Δ.