Lecture 5 (20.3.2018)

(translated and slightly adapted from lecture notes by Martin Klazar)

Theorem 21 (1st Fundamental Theorem of Calculus)**.** *Let $f \in \mathcal{R}(a,b)$ and function $F\colon [a,b] \to \mathbb{R}$ be defined as*

$$F(x) = \int_a^x f \ .$$

Then

(i) F is continuous on $[a,b]$ and

(ii) at every point of continuity $x_0 \in [a,b]$ of f there exists finite derivative $F'(x_0)$ and $F'(x_0) = f(x_0)$ (this applies one-sided if $x_0 = a$ or $x_0 = b$).

Proof. Let $c > 0$ be the upper bound for $|f(x)|$, $a \le x \le b$ (f is integrable and therefore bounded). For every two points $x, x_0 \in [a,b]$ we have

$$|F(x) - F(x_0)| = \Big| \int_a^x f - \int_a^{x_0} f \Big| = \Big| \int_{x_0}^x f \Big| \le |x - x_0| c \ ,$$

according to the definition of F, linearity $\int$ in integration limits and estimate $\int$ by upper sum for a trivial partition of the interval with end points x and x_0. Thus, for $x \to x_0$, we have $F(x) \to F(x_0)$. Therefore, F is continuous in x_0.

Let $x_0 \in [a,b]$ be a point of continuity of f. We have $\delta > 0$ that $f(x_0) - \varepsilon < f(x) < f(x_0) + \varepsilon$ once $|x - x_0| < \delta$. For $0 < x - x_0 < \delta$ then

$$f(x_0) - \varepsilon \le \frac{\int_{x_0}^x f}{x - x_0} = \frac{F(x) - F(x_0)}{x - x_0} \le f(x_0) + \varepsilon \ ,$$

according to the trivial estimate of $\int_{x_0}^x f$ by lower and upper sums for trivial partition (x_0, x). For $-\delta < x - x_0 < 0$ the same inequalities apply (both the numerator and the denominator of the fraction will change sign). For $x \to x_0$, $x \ne x_0$, we have $\frac{F(x) - F(x_0)}{x - x_0} \to f(x_0)$, or $F'(x_0) = f(x_0)$. □

Corollary 22 (Continuous function has a primitive function)**.** *If $f\colon [a,b] \to \mathbb{R}$ is continuous on $[a,b]$, then f has a primitive function F on $[a,b]$.*

Proof. Just use the previous theorem and let $F(x) = \int_a^x f$. □

Theorem 23 (2nd Fundamental Theorem of Calculus)**.** *If $f \in \mathcal{R}(a,b)$ and $F\colon [a,b] \to \mathbb{R}$ is primitive to f on $[a,b]$, then*

$$\int_a^b f = F(b) - F(a) \ .$$

Proof. Let $D = (a_0, a_1, \ldots, a_k)$ be any partition of $[a, b]$. Using Lagrange's mean value theorem for each interval $I_i = [a_{i-1}, a_i]$ and the function F , we get

$$F(b) - F(a) = \sum_{i=1}^{k}(F(a_i) - F(a_{i-1})) = \sum_{i=1}^{k} f(c_i)(a_i - a_{i-1}) ,$$

for some points $a_i < c_i < a_{i+1}$ (since $F'(c_i) = f(c_i)$). Thus, (since $\inf_{I_i} f \leq f(c_i) \leq \sup_{I_i} f$)

$$s(f, D) \leq F(b) - F(a) \leq S(f, D) .$$

Then, from integrability of f, it follows that $F(b) - F(a) = \int_a^b f$. □

For a function $F : [a, b] \to \mathbb{R}$ we denote the difference of functional values in endpoints of the interval by

$$F|_a^b := F(b) - F(a) .$$

Previous results put together yield the following.

Corollary 24 ($\int$ and primitive function)**.** *If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$, then $f \in \mathcal{R}(a, b)$, f has a primitive function F on $[a, b]$ and*

$$\int_a^b f = F|_a^b = F(b) - F(a) .$$

Newton integral.

Let $f : (a, b) \to \mathbb{R}$ be such that a primitive function F of f on (a, b) has one sided limits $F(a^+) = \lim_{x \to a^+} F(x)$ a $F(b^-) = \lim_{x \to b^-} F(x)$. We define Newton integral of f on (a, b) as

$$(N) \int_a^b f = F(b^-) - F(a^+) .$$

Since different primitive functions of f differ by an additive constant, this difference does not depend on the choice of F and the definition is correct. The set of functions which are Newton integrable on (a, b) is denoted by $\mathcal{N}(a, b)$. We denote by $C(a, b)$ the set of functions continuous on $[a, b]$.

Theorem 25 (comparison of Newton and Riemann $\int$)**.**

(i) $C(a, b) \subset \mathcal{N}(a, b) \cap \mathcal{R}(a, b)$.

(ii) If $f \in \mathcal{N}(a, b) \cap \mathcal{R}(a, b)$, then

$$(N) \int_a^b f = (R) \int_a^b f .$$

(iii) The sets $\mathcal{N}(a, b) \backslash \mathcal{R}(a, b)$ and $\mathcal{R}(a, b) \backslash \mathcal{N}(a, b)$ are nonempty.

Proof. If f is continuous on $[a,b]$, by theorem from previous lecture, $f \in \mathcal{R}(a,b)$ and by First fundamental theorem of calculus, $F(x) = \int_a^x f$ is a primitive function to f on $[a,b]$. We have $F(a^+) = F(a) = 0$ a $F(b^-) = F(b) = \int_a^b f$, thus $f \in \mathcal{N}(a,b)$.

Let $f \in \mathcal{N}(a,b) \cap \mathcal{R}(a,b)$. Since $f \in \mathcal{N}(a,b)$, f has a primitive function F on (a,b) with one sided limits $F(a^+)$ and $F(b^-)$. Since $f \in \mathcal{R}(a,b)$, $f \in \mathcal{R}(a+\delta, b-\delta)$ for every $\delta > 0$ and by Second fundamental theorem of calculus we have

$$(R) \int_{a+\delta}^{b-\delta} f = F(b-\delta) - F(a+\delta) \; .$$

For $\delta \to 0^+$ the left hand side tends to $(R) \int_a^b f$ (f is bounded on $[a,b]$, thus integrals of f on $[a, a+\delta]$ and $[b-\delta, b]$ tend to 0) and left hand side tends to $F(b^-) - F(a^+) = (N) \int_a^b f$.

Function $f(x) = x^{-1/2} : \ (0,1] \to \mathbb{R}$, $f(0) = 42$, has Newton integral on $(0,1)$: $F(x) = 2x^{1/2}$ is primitive function of f on $(0,1)$, $F(0^+) = 0$ and $F(1^-) = 2$, thus $(N) \int_0^1 f = 2$. However, f is not bounded on $[0,1]$ and therefore $f \notin \mathcal{R}(0,1)$. Function $\operatorname{sgn}(x)$ is non-decreasing on $[-1,1]$ and thus Riemann integrable on $[-1,1]$. On the other hand, $\operatorname{sgn}(x)$ does not have Newton integral on $(-1,1)$ — as we showed on the first lecture, $\operatorname{sgn}(x)$ does not have a primitive function on $(-1,1)$. □

Next we state variants of methods of computing primitive functions for definite integrals.

Theorem 26 (Integration by parts for definite integral)**.** *Let $f, g : [a,b] \to \mathbb{R}$ be functions with continuous derivatives f' and g' on $[a,b]$. Then,*

$$\int_a^b fg' = fg|_a^b - \int_a^b f'g \; .$$

Theorem 27 (Substitution for definite integral)**.** *Let $\varphi : \ [\alpha,\beta] \to [a,b]$ and $f : \ [a,b] \to \mathbb{R}$ are two functions such that φ has continuous derivative on $[\alpha,\beta]$ and $\varphi(\alpha) = a$, $\varphi(\beta) = b$ or $\varphi(\alpha) = b$, $\varphi(\beta) = a$. If*

(i) f is continous on $[a,b]$, or

(ii) if φ is strictly monotonous on $[\alpha,\beta]$ and $f \in \mathcal{R}(a,b)$

then

$$\int_\alpha^\beta f(\varphi)\varphi' = \int_{\varphi(\alpha)}^{\varphi(\beta)} f = \begin{cases} \int_a^b f \quad \text{or} \\ \\ \int_b^a f = -\int_a^b f \; . \end{cases}$$

Proof of (i). The function f is continuous, so it has a primitive function F. Derivative of a composed function $F(\varphi)$ on $[\alpha,\beta]$ is $F(\varphi)' = f(\varphi)\varphi'$. So, $F(\varphi)$

is on $[\alpha, \beta]$ a primitive function of $f(\varphi)\varphi'$. The function $f(\varphi)\varphi'$ is continuous (since product of two continuous functions is continuous) on $[\alpha, \beta]$, thus, $f(\varphi)\varphi' \in \mathcal{R}(\alpha, \beta)$. Thus, applying 2nd fundamental theorem of calculus twice (the first and the third equality), we have

$$\int_{\alpha}^{\beta} f(\varphi)\varphi' = F(\varphi)|_{\alpha}^{\beta} = F|_{\varphi(\alpha)}^{\varphi(\beta)} = \int_{\varphi(\alpha)}^{\varphi(\beta)} f \ .$$

□