Lecture 4 (13.3.2018)

(translated and slightly adapted from lecture notes by Martin Klazar)

Theorem 14 (Integrability criterion)**.** *Let $f:\ [a,b]\to\mathbb{R}$. Then*

$$f\in\mathcal{R}(a,b) \iff \forall\varepsilon>0\ \exists D:\ 0\le S(f,D)-s(f,D)<\varepsilon.$$

In other words, f has Riemann integral if and only if for every $\varepsilon>0$ there exists a partition of D of interval $[a,b]$ such that its upper Riemann sum is greater than the corresponding lower Riemann sum by less than ε.

Proof. " $\Rightarrow$ " We assume that f has R. integral on $[a,b]$, i.e., $\underline{\int_a^b} f=\overline{\int_a^b} f=\int_a^b f\in\mathbb{R}$. Let $\varepsilon>0$ be given. By definition of the lower and upper integrals, there are partitions E_1 and E_2 so that

$$s(f,E_1)>\underline{\int_a^b} f-\frac{\varepsilon}{2}=\int_a^b f-\frac{\varepsilon}{2}\ \text{ a }\ S(f,E_2)<\overline{\int_a^b} f+\frac{\varepsilon}{2}=\int_a^b f+\frac{\varepsilon}{2}\ .$$

According to the lemma, these inequalities also apply after replacing E_1 and E_2 with their joint refinement $D=E_1\cup E_2$. Summing up both inequalities we will get

$$S(f,D)-s(f,D)<\int_a^b f+\frac{\varepsilon}{2}+\Big(-\int_a^b f+\frac{\varepsilon}{2}\Big)=\varepsilon\ .$$

" $\Leftarrow$ " Given $\varepsilon>0$ we take a partition of D satisfying the condition. According to the definition of the lower and upper integral we get

$$\overline{\int_a^b} f\le S(f,D)<s(f,D)+\varepsilon\le\underline{\int_a^b} f+\varepsilon,\ \text{ thus }\ \overline{\int_a^b} f-\underline{\int_a^b} f<\varepsilon\ .$$

This inequality is valid for every $\varepsilon>0$, so according to the previous statement we have $\overline{\int_a^b} f=\underline{\int_a^b} f\in\mathbb{R}$. Then f has R. integral on $[a,b]$. □

We state another criterion of integrability without a proof.

Theorem 15 (Lebesgue characterisation of integrable functions)**.** *A function $f:\ [a,b]\to\mathbb{R}$ has Riemann integral, if and only if it is bounded and the set of its point of discontinuity on $[a,b]$ has* measure zero*.*

We define sets of measure zero as follows. A set $M\subset\mathbb{R}$ has *(Lebesgue) measure zero*, if for every $\varepsilon>0$, there exists a sequence of intervals $I_1,I_2,\ldots$ such that

$$\sum_{i=1}^{\infty}|I_i|<\varepsilon\ \text{ and }\ M\subset\bigcup_{i=1}^{\infty}I_i\ .$$

In other words, M can be covered by intervals of arbitrarily small length. Simple properties of sets with measure zero:

- Every countable or finite set has measure zero.
- Every subset of a set of measure zero has measure zero.
- If each of countably many sets $A_1, A_2, \ldots$ has measure zero, their union

$$\bigcup_{n=1}^{\infty} A_n$$

has measure zero.
- Interval of positive length does not have measure zero.

For example, the set of rational numbers $\mathbb{Q}$ has measure zero. There exist sets of measure which are uncountable, classical example is *Cantor set*.

Theorem 16 (Monotonicity $\Rightarrow$ integrability)**.** *If $f : [a,b] \to \mathbb{R}$ is non-decreasing or non-increasing on $[a,b]$ then it is Riemann integrable.*

Proof. Assume that f is non-decreasing (for non-increasing f the argument is similar). For each subinterval $[\alpha,\beta] \subset [a,b]$ we have $\inf_{[\alpha,\beta]} f = f(\alpha)$ and $\sup_{[\alpha,\beta]} f = f(\beta)$. Given $\delta > 0$, we take any division $D = (a_0, a_1, \ldots, a_{k-1})$ interval $[a,b]$ with $\lambda(D) < \delta$ and

$$\begin{aligned} S(f,D) - s(f,D) &= \sum_{i=1}^{k}(a_i - a_{i-1})(\sup_{I_i} f - \inf_{I_i} f) \\ &= \sum_{i=1}^{k}(a_i - a_{i-1})(f(a_i) - f(a_{i-1})) \\ &\le \delta \sum_{i=1}^{k}(f(a_i) - f(a_{i-1})) \\ &= \delta(f(a_k) - f(a_0)) = \delta(f(b) - f(a)) \ . \end{aligned}$$

This can be made small by reducing δ, in particular, given ε, choosing $\delta < \varepsilon/(f(b) - f(a))$ ensures that $S(f,D) - s(f,D) < \varepsilon$. According to the integrability criterion, then $f \in \mathcal{R}(a,b)$. □

Continuity is also sufficient for integrability. But we need to introduce its stronger form. Let us say that the function $f : I \to \mathbb{R}$, where I is the interval, is *uniformly continuous* (on I) if

$$\forall\varepsilon > 0\ \exists\delta > 0 :\ x, x' \in I,\ |x - x'| < \delta \Rightarrow |f(x) - f(x')| < \varepsilon\ .$$

That is, we require that single $\delta > 0$ works for all pairs of points x, x' in I. In the usual definition of continuity can δ depend on x. Uniform continuity implies continuity, but the reverse does not generally apply. For example, function

$$f(x) = 1/x : \ I = (0, 1)$$

is continuous on i, but not uniformly continuous: $f(1/(n+1)) - f(1/n) = 1$, although $1/(n+1) - 1/n \to 0$ for $n \to \infty$. On a *compact interval* I, which is the interval of type $[a, b]$ where $-\infty < a \leq b < +\infty$, types of continuity coincide.

Theorem 17 (On compact: continuity $\Rightarrow$ uniform continuity)**.** *If the function* $f : \ [a, b] \to \mathbb{R}$ *on the interval* $[a, b]$ *is continuous, it is uniformly continuous.*

Proof. For contradiction we assume that $f : \ [a, b] \to \mathbb{R}$ is continuous at every point of the interval $[a, b]$ (i.e. one sided in the end points of a a b), but that it is not uniformly continuous to $[a, b]$. Negation of a uniform continuity means, that

$$\exists \varepsilon > 0 \, \forall \delta > 0 \, \exists x, x' \in I : |x - x'| < \delta \,\&\, |f(x) - f(x')| \geq \varepsilon \ .$$

Which means that there are points $x_n, x'_n \in [a, b]$ for $\delta = 1/n$ and $n = 1, 2, \ldots$ that $|x_n - x'_n| < 1/n$, but $|f(x_n) - f(x'_n)| \geq \varepsilon$. Then, by Bolzano–Weierstrass theorem there exist subsequences of (x_n) and (x'_n) which both converge and (inevitably) have the same point α from $[a, b]$. (This theorem asserts that there exists a sequence of indices $k_1 < k_2 < \ldots$ such that (x_{k_n}) converges. Again by the theorem there exists sequence of indices of $l_1 < l_2 < \ldots$ that $(x'_{k_{l_n}})$ converges. The sequence $(x_{k_{l_n}})$ remains convergent, because it is a subsequence of sequence (x_{k_n}). Because $|x_{k_{l_n}} - x'_{k_{l_n}}| < 1/k_{l_n} \leq 1/n \to 0$,

$$\lim_{n \to \infty} x_{k_{l_n}} = \lim_{n \to \infty} x'_{k_{l_n}} = \alpha \ .$$

To avoid multilevel indices, we rename $x_{k_{l_n}}$ to x_n and $x'_{k_{l_n}}$ to x'_n.) By Heine definition of limit, continuity of f in α and arithmetic of limits, we have

$$0 = f(\alpha) - f(\alpha) = \lim f(x_n) - \lim f(x'_n) = \lim(f(x_n) - f(x'_n)) \ .$$

This contradicts that $|f(x_n) - f(x'_n)| \geq \varepsilon$ for every n. $\square$

Theorem 18 (Continuity $\Rightarrow$ integrability)**.** *If* $f : \ [a, b] \to \mathbb{R}$ *on the interval* $[a, b]$ *is continuous then it is Riemann integrable.*

Proof. Let f be continuous on $[a, b]$. Let $\varepsilon > 0$ be given. By the previous statement, we take $\delta > 0$ that $|f(x) - f(x')| < \varepsilon$ when $x, x' \in [a, b]$ are closer than δ. Then

$$\sup_{[\alpha,\beta]} f - \inf_{[\alpha,\beta]} f \leq \varepsilon$$

for each subinterval $[\alpha, \beta] \subset [a, b]$ less than δ (why?). We take any partition $D = (a_0, a_1, \ldots, a_{k-1})$ of interval $[a, b]$ with $\lambda(D)$. We have that

$$S(f, D) - s(f, D) \;\; = \;\; \sum_{i=1}^{k} (a_i - a_{i-1})(\sup_{I_i} f - \inf_{I_i} f)$$

$$
\begin{aligned}
&\le \sum_{i=1}^{k}(a_i - a_{i-1})\varepsilon \\
&= \varepsilon(a_k - a_0) = \varepsilon(b-a) \ .
\end{aligned}
$$

As in the previous theorem, the $\varepsilon(b-a)$ can be made small by reducing ε. Thus, according to the integrability criterion, $f \in \mathcal{R}(a,b)$. □

Theorem 19 (Linearity of Riemann integral)**.**

(i) (linearity w.r.to integrand) Let $f, g \in \mathcal{R}(a,b)$ be two functions having R. integrals and $\alpha, \beta \in \mathbb{R}$. Then

$$\alpha f + \beta g \in \mathcal{R}(a,b) \ \textit{and} \ \int_a^b (\alpha f + \beta g) = \alpha \int_a^b f + \beta \int_a^b g \ .$$

(ii) (linearity w.r. to boundaries) Let $f : [a,b] \to \mathbb{R}$ be a function and $c \in (a,b)$. Then

$$f \in \mathcal{R}(a,b) \iff f \in \mathcal{R}(a,c) \ \& \ f \in \mathcal{R}(c,b)$$

and, if these integrals are defined,

$$\int_a^b f = \int_a^c f + \int_c^b f \ .$$

Proof. (i) Just examine three special cases of linear combinations, namely $-f$, αf s $\alpha \ge 0$ and $f+g$, the others are from these are already deduced. Either given $\varepsilon > 0$. According to the integrity criterion, there is a division of the D interval $[a,b]$ that

$$S(f,D) - s(f,D), S(g,D) - s(g,D).$$

(Surely we have two such divisions, D_1 for f and D_2 for g. Moving to a common refinement, we will achieve $D_1 = D_2$.) By definition of infima and suprema of a set of real numbers, for any subinterval $I \subset [a,b]$, that (pro $\alpha \ge 0$)

$$\inf_I(-f) = -\sup_I f, inf_I \alpha f = \alpha \inf_I f, inf_I(f+g) \ge \inf_I f + \inf_I g$$

and for supremacy (we will swap inf a sup and rotate the last inequality). By definition, upper, sums as a linear combination (with > 0 coefficients) infim, or supreme,

$$S(-f,D) - s(-f,D) = -s(f,D) - (-S(f,D)) = S(f,D) - s(f,D) < \varepsilon \ ,$$

and

$$\begin{aligned} S(f+g,D)-s(f+g,D) &\le (S(f,D)+S(g,D))-(s(f,D)+s(g,D)) \\ &= S(f,D)-s(f,D)+S(g,D)-s(g,D) \\ &< 2\varepsilon\ . \end{aligned}$$

So, according to the integrity criterion, i $-f, \alpha f, f+g \in \mathcal{R}(a,b)$. Moreover, according to the inequalities between the lower and upper sums and the integral, $\int_a^b f \in [s(f,D), S(f,D)]$ and the same for g. So $\int_a^b(-f)$ lies in the interval

$$[s(-f,D), S(-f,D)] = [-S(f,D), -s(f,D)] \ni -\int_a^b f$$

and the numbers $\int_a^b(-f)$ a $-\int_a^b f$ differ by less than ε. So $\int_a^b(-f) = -\int_a^b f$. Similarly $\int_a^b \alpha f$ lie in the interval

$$[s(\alpha f,D), S(\alpha f,D)] = [\alpha s(f,D), \alpha S(f,D)] \ni \alpha\int_a^b f$$

with a maximum length of $\alpha\varepsilon$, so $\int_a^b \alpha f = \alpha\int_a^b f$. Finally, $\int_a^b(f+g)$ is in the interval

$$[s(f+g,D), S(f+g,D)] \subset [s(f,D)+s(g,D), S(f,D)+S(g,D)] \ni \int_a^b f + \int_a^b g$$

with a length of less than 2ε, and so $\int_a^b(f+g) = \int_a^b f + \int_a^b g$.

(ii) Let's go to line $\int$ as a function of integration limits. First, we slightly extend the definition of $\int_a^b f$:

$$\int_a^a f := 0 a int_a^b f := -\int_b^a f mbox pro a > b$$

For $f:\ [a,b] \to \mathbb{R}$ and subinterval $I \subset [a,b]$ we denote the f function narrowing to I in the following statement for simplicity again as f.

□

If $a > b$, we define $\int_a^b f = -\int_b^a f$.

Corollary 20 ($\int$ over a cycle is 0)**.** *Let* $a,b,c \in \mathbb{R}$*,* $d = \min(a,b,c)$*,* $e = \max(a,b,c)$ *and* $f \in \mathcal{R}(d,e)$*. Then the following three integrals exist and satisfy*

$$\int_a^b f + \int_b^c f + \int_c^a f = 0\ .$$