Lecture 2 (27.2.2019)

(translated and slightly adapted from lecture notes by Martin Klazar)
(Warning: not a substitute for attending the lectures, probably contains typos. let me know if you spot any!)

Methods for computing primitive functions

To calculate the derivative of the product of two functions, we have the Leibniz formula $(fg)' = f'g + fg'$. By inverting it, we will get the following important result for primitive functions.

Theorem 6 (Integration per partes (by parts))**.** *If $f, g : I \to \mathbb{R}$ are continuous functions on an interval I and F, G their corresponding primitive functions on I then the following equality holds on I:*

$$\int fG + \int Fg = FG + c\ .$$

In other words, the functions fG and Fg have primitive functions on I whose sum is always equal to function FG on I, up to the additive constant c.

Proof. Since that f and g are continuous on I, and by Theorem 3, the primitive functions F and G are also continuous. So products fG and Fg are also continuous, and by Theorem 4, they have primitive functions $\int fG$ and $\int Fg$ on I. By linearity of primitive functions, the sum $\int fG + \int Fg$ is a primitive function of $fG + Fg$. Moreover, the FG function is a primitive function of $fG + Fg$, because the Leibniz formula gives $(FG)' = fG + Fg$. Thus, we get that $\int fG + \int Fg = FG + c$. $\square$

The formula for integration per partes is usually given in an equivalent form

$$\int F'G = FG - \int FG'$$

So if we can calculate the primitive function of FG' for the two functions F and G with continuous derivatives ($F' = f$ and $G' = g$) we get a primitive function of $F'G$ according to this formula.

Example 2. *With $x' = 1$ and $(\log x)' = 1/x$ on the interval $(0, +\infty)$ we have*

$$\int \log x = \int x' \log x = x \log x - \int x(\log x)' = x \log x - \int 1 = x \log x - x + c$$

on $(0, +\infty)$. By taking derivative, we can easily check the correctness of the derived formula.

Inverting the rule for derivative of the product gives the formula for integration per partes and by inverting the rule for derivative of the composed function we get a formula for integration by substitution. It has two forms, according to the direction of reading the equality of $f(\varphi)' = f'(\varphi)\varphi'$.

Theorem 7 (Integration by substitution)**.** *Let* $\varphi : (\alpha, \beta) \to (a, b)$ *and* $f : (a, b) \to \mathbb{R}$ *be two functions such that* φ *has a proper derivative* φ' *on* (α, β).

1. *If* $F = \int f$ *on* (a, b), *then* $\int f(\varphi)\varphi' = F(\varphi) + c$ *on* (α, β).
2. *Suppose* φ *additionally that* $\varphi((\alpha, \beta)) = (a, b)$ *and either* $\varphi' > 0$ *or* $\varphi' < 0$ *on* (α, β). *If* $G = \int f(\varphi)\varphi'$ *on* (α, β), *then* $\int f = G(\varphi^{\langle -1 \rangle}$ *on* (a, b).

Proof. The first part follows immediately by the derivative:

$$F(\varphi)' = F'(\varphi)\varphi' = f(\varphi)\varphi'$$

on (α, β), from the assumption about F and derivative of the composed function.

In the second part assumptions about φ guarantee that it is a strictly increasing or a strictly decreasing bijection from (α, β) to (a, b). So it is an injective function, it has an inverse function

$$\varphi^{\langle -1 \rangle} : (a, b) \to (\alpha, \beta).$$

We can compute derivative of this function using the inverse function derivative rule. This gives, together with the assumption about G, derivative of the composed function and the derivative of the inverse function, that $G(\varphi^{\langle -1 \rangle})$ is primitive function of f on (a, b) :

$$G(\varphi^{\langle -1 \rangle})' = G'(\varphi^{\langle -1 \rangle}) \cdot (\varphi^{\langle -1 \rangle})' = f(\varphi(\varphi^{\langle -1 \rangle}))\varphi'(\varphi^{\langle -1 \rangle}) \cdot \frac{1}{\varphi'(\varphi^{\langle -1 \rangle})} = f \ .$$

□

Here are two examples of both forms of the substitution rule.

1. When $F(x) = \int f(x)\, dx$ at some I a $a, b \in \mathbb{R}$, $a \neq 0$, then according to the first part we calculate that

$$\int f(ax + b)\, dx = a^{-1} \int f(ax + b) \cdot (ax + b)'\, dx = a^{-1} F(ax + b) + c \ ,$$

on the interval $J = a^{-1}(I - b) = \{a^{-1}(x - b) \mid x \in I\}$. It is easy to check backwards by taking derivative. We took $\varphi(x) = ax + b$.

2. We want to calculate the primitive function of $\sqrt{1 - t^2}$ on $(-1, 1)$. Because it resembles the derivative of arcsin, we try the substitution $t = \varphi(x) = \sin x : (-\frac{\pi}{2}, \frac{\pi}{2}) \to (-1, 1)$. The assumptions of the second form of the substitution rule are fulfilled.

$$G(x) = \int \sqrt{1-\sin^2 x} \cdot (\sin x)' \, dx = \int \sqrt{\cos^2 x} \cdot \cos x \, dx = \int \cos^2 x \, dx \, .$$

Did it help? It helped because the last primitive function can easily be calculated by integrating per partes:

$$\begin{aligned} \int \cos^2 x &= \int \cos x (\sin x)' = \cos x \sin x + \int \sin^2 x \\ &= \cos x \sin x + \int (1 - \cos^2 x) \\ &= \sin x \cos x + x - \int \cos^2 x \, , \end{aligned}$$

so,

$$G(x) = \int \cos^2 x = \frac{\sin x \cos x + x}{2} + c = \frac{\sin x \sqrt{1 - \sin^2 x} + x}{2} + c \, .$$

After letting $x = \varphi^{\langle -1 \rangle}(t) = \arcsin t$ we get the desired result

$$\int \sqrt{1-t^2} = G(\arcsin t) + c = \frac{t\sqrt{1-t^2} + \arcsin t}{2} + c, \quad \text{on } (-1, 1) \, .$$

By derivative, we can easily verify it is correct.

By saying that *f can be expressed using elementary functions* we mean that f can be expressed from the basic functions $\exp(x)$ (exponential), $\log x$, $\sin x$, $\arcsin x$, $\cos x$, $\arccos x$, $\tan x$ and $\arctan x$ repeatedly using the arithmetic operations $+, -, \times, :$, and the folding operations. Many primitive functions can be expressed in this way, but many primitive functions cannot. The following theorem, which we will not prove, gives some important examples of such functions.

Theorem 8 (Non-elementary primitive functions)**.** *Primitive functions*

$$F_1(x) = \int \exp(x^2), \;\; F_2(x) = \int \frac{\sin x}{x} \quad \textit{and} \quad F_3(x) = \int \frac{1}{\log x}$$

(on the intervals where they are defined) cannot be expressed using elementary functions.

Primitive functions of rational functions

A relatively wide class of functions to which primitive functions can be computed are *rational functions*, which are fractions of polynomials. Let's give a simple example. Let $I \subset \mathbb{R}$ be any open interval that does not contain -1 and 1. Then

$$\begin{aligned}
\int \frac{x^2}{x^2-1} &= \int \Big(1+\frac{1}{x^2-1}\Big) = \int \Big(1+\frac{1/2}{x-1}-\frac{1/2}{x+1}\Big) \\
&= \int 1 + \frac{1}{2}\int \frac{1}{x-1} - \frac{1}{2}\int \frac{1}{x+1} \\
&= x + \frac{\log|x-1| - \log|x+1|}{2} + c \\
&= x + \log(\sqrt{|(x-1)/(x+1)|}) + c
\end{aligned}$$

on I . It turns out that similarly, a primitive function can be calculated for any rational function. The key is a decomposition to the sum simpler rational functions (the first line of calculation), which is called decomposition into *partial fractions*. In the following we use some results from algebra that we will not prove here.

Theorem 9 (Primitive function for rational function can always be calculated)**.** *Let $P(x)$ and $Q(x) \neq 0$ be polynomials with real coefficients and $I \subset \mathbb{R}$ is an open interval not containing no roots of $Q(x)$. Primitive function*

$$F(x) = \int \frac{P(x)}{Q(x)} \qquad \textit{(on } I\textit{)}$$

can be expressed using elementary functions, namely using rational functions, logarithms and arcustangent.

Proof. Without loss of generality, assume that $Q(x)$ is monic (i.e. its leading coefficient is 1). After dividing $P(x)$ by $Q(x)$ with remainder we have

$$\frac{P(x)}{Q(x)} = p(x) + \frac{R(x)}{Q(x)} \ ,$$

where $p(x), R(x)$ are real polynomials and $R(x)$ has smaller degree than $Q(x)$. There is unique way to express $Q(x)$ as a product of irreducible real polynomials (i.e. polynomials that cannot be expressed as product of polynomials of smaller degree), moreover, these polynomials will have degree at most 2:

$$Q(x) = \prod_{i=1}^{k} (x-\alpha_i)^{m_i} \prod_{i=1}^{l} (x^2+\beta_i x+\gamma_i)^{n_i} \ ,$$

where $k, l \geq 0$ are integers, $\alpha_i, \beta_i, \gamma_i \in \mathbb{R}$, $m_i, n_i \geq 1$ are integers, numbers α_i are pairwise distinct, pairs (β_i, γ_i) are pairwise distinct and always $\beta_i^2 - 4\gamma_i < 0$ (thus, the polynomial $x^2+\beta_i x+\gamma_i$ is irreducible as it has no real roots). It can be shown that $R(x)/Q(x)$ has unique expression as the sum of *partial fractions*

$$\frac{R(x)}{Q(x)} = \sum_{i=1}^{k}\sum_{j=1}^{m_i} \frac{\delta_{i,j}}{(x-\alpha_i)^j} + \sum_{i=1}^{l}\sum_{j=1}^{n_i} \frac{\epsilon_{i,j}x+\theta_{i,j}}{(x^2+\beta_i x+\gamma_i)^j} \ ,$$

where $\delta_{i,j}, \epsilon_{i,j}, \theta_{i,j} \in \mathbb{R}$. In the previous example we have $P(x) = x^2$, $Q(x) = x^2 - 1$, $p(x) = 1$, $R(x) = 1$, $k = 2$, $m_1 = m_2 = 1$, $l = 0$ (decomposition of $Q(x)$ contains no quadratic polynomial with three non-zero coefficients), $\alpha_1 = 1$, $\alpha_2 = -1$, $\delta_{1,1} = \frac{1}{2}$ a $\delta_{2,1} = -\frac{1}{2}$. Thus, a primitive function $\int \frac{P(x)}{Q(x)}$ equals to the sum of finitely many primitive functions of three types:

$$\int p(x), \int \frac{\delta}{(x-\alpha)^j} \quad \text{a} \quad \int \frac{\epsilon x + \theta}{(x^2 + \beta x + \gamma)^j} ,$$

where $p(x)$ is a real polynomial, $j \in \mathbb{N}$ and except x all other symbols are real constants, and $\beta^2 - 4\gamma < 0$. If we can express these primitive functions using elementary functions, we can express $\int \frac{P(x)}{Q(x)}$ using elementary functions as well.

It is easy to calculate primitive functions of the first two types:

$$\int p(x) = \int (a_n x^n + \ldots + a_1 x + a_0) = \frac{a_n x^{n+1}}{n+1} + \ldots + \frac{a_1 x^2}{2} + a_0 x$$

on $\mathbb{R}$ and

$$\int \frac{\delta}{(x-\alpha)^j} = \frac{\delta}{(1-j)(x-\alpha)^{j-1}} \ (j \geq 2), \ \int \frac{\delta}{x-\alpha} = \delta \log |x - \alpha|$$

on $(-\infty, \alpha)$ and $(\alpha, +\infty)$ (we omitted additive constants). The third type is more complex. We have

$$\int \frac{\epsilon x + \theta}{(x^2 + \beta x + \gamma)^j} = \frac{\epsilon}{2} \int \frac{2x + \beta}{(x^2 + \beta x + \gamma)^j} + (\theta - \epsilon\beta/2) \int \frac{1}{(x^2 + \beta x + \gamma)^j} .$$

For the last but one $\int$ is after substituting $y = x^2 + \beta x + \gamma$ of the second typewe have

$$\int \frac{2x + \beta}{(x^2 + \beta x + \gamma)^j} = \frac{1}{(1-j)(x^2 + \beta x + \gamma)^{j-1}} \ (j \geq 2)$$

and

$$\int \frac{2x + \beta}{x^2 + \beta x + \gamma} = \log |x^2 + \beta x + \gamma| = \log(x^2 + \beta x + \gamma) .$$

on $\mathbb{R}$ (recall that $x^2 + \beta x + \gamma$ has no real root) . It remains to calculate a primitive function $\int 1/(x^2 + \beta x + \gamma)^j$. We denote $\eta = \sqrt{\gamma - \beta^2/4}$ (recall that $\gamma - \beta^2/4 > 0$) and use substitution $y = y(x) = x/\eta + \beta/2\eta$. By completing the square we get

$$\begin{aligned} \int \frac{1}{(x^2 + \beta x + \gamma)^j} &= \frac{1}{\eta^{2j-1}} \int \frac{1/\eta}{((x/\eta + \beta/2\eta)^2 + 1)^j} \\ &= \frac{1}{\eta^{2j-1}} \int \frac{y'}{((x/\eta + \beta/2\eta)^2 + 1)^j} \\ &= \frac{1}{\eta^{2j-1}} \int \frac{1}{(y^2 + 1)^j} . \end{aligned}$$

Thus, it remains to compute the following primitive function on $\mathbb{R}$:

$$I_j = \int \frac{1}{(1+x^2)^j} \ .$$

For $j = 1$ we already know that $I_1 = \arctan x$. For $j = 2, 3, \ldots$ we express I_j using recurrence obtained by integration by parts:

$$\begin{aligned}
I_j &= \int \frac{x'}{(1+x^2)^j} = \frac{x}{(1+x^2)^j} + \int \frac{2jx^2}{(1+x^2)^{j+1}} \\
&= \frac{x}{(1+x^2)^j} + 2j \int \frac{x^2+1}{(1+x^2)^{j+1}} - 2j \int \frac{1}{(1+x^2)^{j+1}} \\
&= \frac{x}{(1+x^2)^j} + 2jI_j - 2jI_{j+1} \ ,
\end{aligned}$$

thus

$$I_{j+1} = I_j(1 - 1/2j) + \frac{x}{2j(1+x^2)^j} \ .$$

For instance,

$$I_2 = \frac{\arctan x}{2} + \frac{x}{2(1+x^2)} \ \text{ a } \ I_3 = \frac{3\arctan x}{8} + \frac{3x}{8(1+x^2)} + \frac{x}{4(1+x^2)^2} \ .$$

In general, the recurrence shows that for every $j = 1, 2, \ldots$, I_j has form $I_j = \kappa \arctan x + r(x)$, where κ is a fraction and $r(x)$ is a rational function. Thus, we have completed the calculation of the primitive function of the third type from the expression $R(x)/Q(x)$ of the sum of the partial fractions and obtained a complete expression of the primitive function $\int \frac{P(x)}{Q(x)}$ using elementary functions. $\square$

Riemann integral

Now we define precisely the concept of the area, in particular, the area of figure $U(a, b, f)$ under the graph of a function f. Let $-\infty < a < b < +\infty$ be two real numbers and $f : [a, b] \to \mathbb{R}$ any function that may not be continuous or bounded. The finite $k+1$-tuple of points $D = (a_0, a_1, \ldots, a_k)$ from the interval $[a, b]$ is called a *partition* of $[a, b]$ if

$$a = a_0 < a_1 < a_2 < \ldots < a_k = b \ .$$

These points divide the interval $[a, b]$ into intervals $I_i = [a_{i-1}, a_i]$. We denote by $|I_i|$ the length of interval I_i: $|I_i| = a_i - a_{i-1}$ a $|[a, b]| = b - a$. Clearly

$$\sum_{i=1}^{k} |I_i| = (a_1 - a_0) + (a_2 - a_1) + \ldots + (a_k - a_{k-1}) = b - a = |[a, b]| \ .$$

Norm of a partition D is the maximum length of an interval of the partition and is denoted by λ:

$$\lambda = \lambda(D) = \max_{1 \le i \le k} |I_i| \ .$$

Partition of an interval $[a, b]$ *with points* is a pair (C, D) wher $D = (a_0, a_1, \ldots, a_k)$ is a partition of $[a, b]$ and a k-tuple $C = (c_1, c_2, \ldots, c_k)$ consists of $c_i \in I_i$ (i.e. $a_{i-1} \le c_i \le a_i$). *Riemann sum* corresponding to the function f and a partition with points (D, C) is defined as

$$R(f, D, C) = \sum_{i=1}^{k} |I_i| f(c_i) = \sum_{i=1}^{k} (a_i - a_{i-1}) f(c_i) \ .$$

If $f \ge 0$ on $[a, b]$, it is the sum of k rectangles $I_i \times [0, f(c_i)]$ whose union approximates figure $U(a, b, f)$. However, Riemann sum is defined for every function f, regardless of its sign on $[a, b]$. The following definition was introduced by Bernhard Riemann (1826–1866).

Definition 2 (First definition of Riemann integral, Riemann)**.** *We say that* $f : [a, b] \to \mathbb{R}$ *has Riemann integral* $I \in \mathbb{R}$ *on* $[a, b]$ *if for every* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that for each partition of* $[a, b]$ *with points* (D, C) *such that* $\lambda(D) < \delta$ *the following holds:*

$$|I - R(f, D, C)| < \varepsilon$$

Therefore, we require $I \in \mathbb{R}$, values $\pm\infty$ are not allowed (although, it is possible to define them). If there is such a number I, we write

$$I = \int_a^b f(x)\, dx = \int_a^b f = (\mathcal{R}) \int_a^b f$$

and say that f is *Riemann integrable* on the interval $[a, b]$. We will work with the class of all Riemann integrable functions

$$\mathcal{R}(a, b) := \{f \mid f \text{ is defined and Riemann integrable on } [a, b]\}.$$

Thus, the first definition of the Riemann integral can be summarized by the formula

$$\int_a^b f = \lim_{\lambda(D) \to 0} R(f, D, C) \in \mathbb{R} \ .$$

We understand the limit here as defined in the definition above; as a symbol, we defined only limit of a sequence and of a function in a point.

For the second, equivalent, definition of the integral we will need a few more concepts. For $f : [a, b] \to \mathbb{R}$ and a partition $D = (a_0, a_1, \ldots, a_k)$ of interval $[a, b]$ we define *lower and upper Riemann sum*, respectively, (even though they were introduced by Darboux) as

$$s(f, D) = \sum_{i=1}^{k} |I_i| m_i, \ \text{ and } \ S(f, D) = \sum_{i=1}^{k} |I_i| M_i \ ,$$

where

$$m_i = \inf_{x \in I_i} f(x) \text{ and } \; M_i = \sup_{x \in I_i} f(x)$$

$$I_i = [a_{i-1}, a_i]$$

These sums are always defined $s(f, D) \in \mathbb{R} \cup \{-\infty\}$ and $S(f, D) \in \mathbb{R} \cup \{+\infty\}$ *Lower and upper Riemann integral*, respectively, of a function f on the interval $[a, b]$ is defined as

$$\underline{\int_a^b} f = \underline{\int_a^b} f(x)\, dx = \sup(\{s(f, D) : \; D \text{ is a partition of } [a, b]\}) \; ,$$

and

$$\overline{\int_a^b} f = \overline{\int_a^b} f(x)\, dx = \inf(\{S(f, D) : \; D \text{ is a partition of } [a, b]\}) \; .$$

These terms are always defined and we have $\underline{\int_a^b} f, \overline{\int_a^b} f \in \mathbb{R}^* = \mathbb{R} \cup \{-\infty, +\infty\}$.

Definition 3 (Second definition of Riemann integral, Darboux)**.** *We say that* $f : \; [a, b] \to \mathbb{R}$ *has at* $[a, b]$ *Riemann integral, if*

$$\underline{\int_a^b} f(x)\, dx = \overline{\int_a^b} f(x)\, dx \in \mathbb{R} \; .$$

This common value, if it exists, we denote by

$$\int_a^b f(x)\, dx = \int_a^b f$$

and we call it the Riemann integral of f *on the interval* $[a, b]$*.*

The two definitions are equivalent: they give the same classes of Riemann integrable functions and the same value of the Riemann integral, if defined.