Lecture 12 (22.5.2019)

(translated and slightly adapted from lecture notes by Martin Klazar)

Metric and topological spaces

Metric space is a structure formalizing distance. It is a pair (M, d) consisting of $M \neq \emptyset$ and a function of two variables

$$d: \ M \times M \to \mathbb{R},$$

called a *metric*, which satisfies the following three axioms:

- $d(x, y) \geq 0$ (non-negativity) a $d(x, y) = d(y, x)$ (symmetry),
- $d(x, y) = 0 \iff x = y$ and
- $d(x, y) \leq d(x, z) + d(z, y)$ (triangle inequality).

The non-negativity of the metric in a) does not have to be required, it follows from axioms b) and c). Here are some examples of metric spaces. Axioms a) and b) can usually be checked easily. Proving triangle inequality is often more difficult.

Example 5. *$M = \mathbb{R}^n$ a $p \geq 1$ is a real number. At M we define $d_p(x, y)$ metrics*

$$d_p(x, y) = \left(\sum_{i=1}^{n} |x_i - y_i|^p \right)^{1/p}$$

($x = (x_1, x_2, \ldots, x_n), y = (y_1, y_2, \ldots, y_n)$). For $n = 1$ we get classical metrics $|x - y|$ to $\mathbb{R}$ and for $p = 2, n \geq 2$ Euclidean metrics

$$d_2(x, y) = \|x - y\| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}.$$

For $p = 1, n \geq 2$ we get Manhattan metric

$$d_1(x, y) = \sum_{i=1}^{n} |x_i - y_i|$$

and for $p \to \infty$ maximum metric

$$d_\infty(x, y) = \max_{1 \leq i \leq n} |x_i - y_i| \, .$$

Example 6. *For M we take a set of all bounded functions $f: \ X \to \mathbb{R}$ defined on the X set. At M then we have* supremum metric

$$d(f, g) = \sup_{x \in X} |f(x) - g(x)| \, .$$

Example 7. *For a connected graph $G = (M, E)$ with a set of vertices M, we have a metric*

$d(u, v) =$ *the number of edges on the shortest path in G joining vertices u and v*

Example 8. *Let A be a set (alphabet) and let $M = A^m$ be the set of strings of length m over the alphabet A ($u = a_1 a_2 \dots a_m, v = b_1 b_2 \dots b_m$). So called* Hamming Metric

$$d(u, v) = \textit{number of coordinates } i\textit{, for which } a_i \neq b_i \ .$$

It measures the degree of difference between the two words, i.e., the smallest number of changes in the letters needed for converting u into v.

We will introduce a few basic concepts; with many we have already met in Euclidean spaces. Let (M, d) be a metric space. Then

- *(open) ball* in M with centre $a \in M$ and radius $\mathbb{R} \ni r > 0$ is the set $B(a, r) = \{x \in M \mid d(a, x) < r\}$;
- $A \subset M$ is *open set* if $\forall a \in A \ \exists r > 0 : \ B(a, r) \subset A$;
- $A \subset M$ is a *closed set* if $M \backslash A$ is an open set;
- $A \subset M$ is a *bounded set* if there is a point $a \in M$ and a radius $r > 0$ that $A \subset B(a, r)$;
- $A \subset M$ is a *compact set* if each sequence of points $(a_n) \subset A$ has a convergent subsequence, whose limit lies in A.

Convergence and limit are generalized from the real axis to the general metric space in an obvious way: sequence $(a_n) \subset M$ is *convergent and has a limit* $a \in M$, (we write $\lim_{n\to\infty} a_n = a$) when

$$\forall \varepsilon > 0 \ \exists n_0 : \ n > n_0 \Rightarrow d(a_n, a) < \varepsilon$$

In other words, $\lim_{n\to\infty} d(a_n, a) = 0$ (we have converted it to the real sequence limit).

We have already mentioned the properties of open sets: $\emptyset$ and M are open, union of any system sets of open sets is an open set, and the intersection of any finite system of open sets is an open set. By switching to the complement, we have the dual properties of closed sets: $\emptyset$ and M are closed, the union of any finite system of closed sets is a closed set, and the intersection of any set system of closed sets is a closed set.

Theorem 51 (Characterisation of closed sets)**.** *A set $A \subset M$ is closed in M, if and only if the limit of every convergent sequence $(a_n) \subset A$ belongs to A.*

Proof. Let $A \subset M$ be a closed set and $(a_n) \subset A$ a convergent sequence. If $\lim_{n\to\infty} a_n = a \notin A$, there exists a radius $r > 0$ such that $B(a, r) \subset M\backslash A$. But then $d(a_n, a) \geq r$ for every n, this contradicts that $\lim_{n\to\infty} a_n = a$. So $a \in A$.

Conversely, if the $A \subset M$ is not a closed set, there is a point $a \in M\backslash A$ such that for each radius $r > 0$ is $B(a, r) \cap A \neq \emptyset$. We put $r = 1/n$, $n = 1, 2, \ldots$, and for each n choose a point $a_n \in B(a, 1/n) \cap A$. Then $(a_n) \subset A$ is a convergent sequence with $\lim_{n\to\infty} a_n = a$, but $a \notin A$. □

Topological spaces. Topological spaces are generalization of metric spaces. The pair $T = (X, \mathcal{T})$, where X is the set and $\mathcal{T}$ is a system of its subsets is a topological space if $\mathcal{T}$ has the following properties:

(i) $\emptyset, X \in \mathcal{T}$,

(ii) $\bigcup \mathcal{U} \in \mathcal{T}$ for every subsystem $\mathcal{U} \subset \mathcal{T}$, and

(iii) $\bigcap \mathcal{U} \in \mathcal{T}$ for every finite subsystem $\mathcal{U} \subset \mathcal{T}$.

Sets in the $\mathcal{T}$ system is called the *open sets* of the topological space T (their complements to X are then *closed sets* of the T space). Example of topological space are the open sets of each metric space. However, there are plenty of topological spaces, which are not metrizable (i.e. do not come from metric space).

Continuous mappings. Let (M, d) and (N, e) be two metric spaces. We say that a mapping

$$f :\ M \to N$$

is *continuous*, if

$$\forall a \in M, \varepsilon > 0\ \exists \delta > 0 :\ b \in M, d(a, b) < \delta \Rightarrow e(f(a), f(b)) < \varepsilon\ .$$

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Theorem 52 (Topological definition of continuity)**.** *A mapping $f :\ M \to N$ between metric spaces is continuous, if and only if for every open set $B \subset N$ is its preimage $f^{-1}(B) = \{x \in M \mid f(x) \in B\}$ open set in M.*

Theorem 53 (Compact $\Rightarrow$ closed and bounded)**.** *Each compact set in the metric space is closed and bounded.*

Proof. Let $A \subset M$ be a subset in metric space (M, d). When A is not closed, there is convergence the sequence $(a_n) \subset A$, whose limit a does not belong to A. Each subsequence of (a_n) is also convergent and has the same limit a. This means that no subsequence (a_n) is has its limit within A (the limit is determined unambiguously) and thus A is not compact.

When A is not bounded, it is not contained in any $B(a, r)$ balls and we can easily build a sequence $(a_n) \subset A$ with the property that $d(a_m, a_n) \geq 1$ for every two indices $1 \leq m < n$. This property contradicts sequence convergence (why?) every subsequence of (a_n)has this property , so (a_n) has no convergent subsequence. A is not compact again.

We define a sequence $(a_n) \subset A$ with the specified property inductively. We take the first point $a_1 \in A$ arbitrarily. Assume that we have already constructed points $a_1, a_2, \dots, a_k$ of A, such that the distance of each pair is at least 1. Then we take any sphere $B(a, r)$, which contains all of these points (each finite set is bounded) and consider the $B(a, r+1)$ sphere. Since A is not bounded, there exists point $a_{k+1} \in A$ that is not in $B(a, r+1)$. According to the triangle inequality, $d(a_{k+1}, x) \geq 1$ for every point $x \in B(a, r)$ (why?). Thus a_{k+1} has distance at least 1 from each point $a_1, a_2, \dots, a_k$ a $a_1, a_2, \dots, a_k$ we can extend to $a_1, a_2, \dots, a_k, a_{k+1}$. Thus defined sequence a_k, $k = 1, 2, \dots$ has the required property. □

Probably the simplest example showing that the converse does not hold in general is the following. Let (M, d) be a trivial metric space, where $d(x, y) = 1$ for $x \neq y$ a $d(x, x) = 0$ (verify that this is a metric space), and the M set is infinite. Then each the sequence $(a_n) \subset M$, where a_n are mutually different points (for the existence of such a sequence we need infinity M) satisfies that $d(a_m, a_n) \geq 1$ for every two indices $1 \leq m < n$. As we know, such a sequence has no convergent subsequence and therefore M is not a compact set. But M is a closed set and it is also bounded because it is a subset of $B(a, 2)$ for any point $a \in M$.

As we have already mentioned, the converse holds for the Euclidean spaces.

Theorem 54 (Closed and bounded $\Rightarrow$ compact in $\mathbb{R}^k$). *Each closed and bounded set in the Euclidean space $\mathbb{R}^k$ is compact.*

Theorem 55 (Continuous function attains extremes on compact). *Let $f : M \to \mathbb{R}$ be a continuous function from the metric space (M, d) into the Euclidean space $\mathbb{R}^1$ and M is compact. Then f has minimum and maximum on M.*