**Lecture 1 (20.2.2019)**

(translated and slightly adapted from lecture notes by Martin Klazar)
(Warning: not a substitute for attending the lectures, probably contains typos. Let me know if you spot any!)

# Primitive functions

**Definition 1** (Primitive function)**.** *If $I \subseteq \mathbb{R}$ is a non-empty open interval and*

$$F, f : I \to \mathbb{R}$$

*are functions satisfying $F' = f$ on $I$, we call $F$ a* primitive function of *$f$ on $I$.*

First, some motivation — the relation of primitive functions with planar figures. For a nonnegative and continuous function $f : [a, b] \to \mathbb{R}$ we consider a planar figure

$$U(a, b, f) = \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b \ \& \ 0 \leq y \leq f(x)\} .$$

Its area, whatever it is, is denoted by

$$\int_a^b f := \text{area}(U(a, b, f)) .$$

This is the area of part of the plane defined by the axis $x$, graph of the function $f$ and the vertical lines $y = a$ and $y = b$. Two basic relationships between the area and the derivative are as follows. Consider the function $F$ of $x$ defined as the area $U(a, x, f)$, i.e., $F(x) = \int_a^x f$. *The first fundamental theorem of calculus* says that for every $c \in [a, b]$ we have

$$F'(c) = f(c)$$

— the derivative of a function whose argument $x$ is the upper limit of the $U(a, x, f)$ and the value is its area is equal to the original function $f$. Thus, the function $F(x) = \int_a^x f$ is a primitive function of $f$. According to the *second fundamental theorem of calculus* for every function $F$, which is a primitive function of $f$ on $[a, b]$, it holds that

$$\int_a^b f = F(b) - F(a) .$$

If we know a primitive function of $f$ (many can be deduced by simply reversing the rules for derivative of elementary functions), we can immediately calculate the area of $U(a, b, f)$. We formulate and prove both theorems precisely later in the lecture on Riemann's integral, when we also introduce the area $\int_a^b f$. But first we need to look at the properties of primitive functions — where the function has a primitive function, whether it is unique, etc.

Due to the linearity of the derivative, primitive functions are also linear:

**Theorem 1** (Linearity of primitive functions)**.** *If $F$ is a primitive function of $f$, and $G$ is a primitive function of $g$ on an interval $I$ and $\alpha, \beta \in \mathbb{R}$, then the function*

$$\alpha F + \beta G$$

*is a primitive function of $\alpha f + \beta g$ on $I$.*

For limit and derivative, the result of the operation is unique if it exists, but a primitive function is not unique. We will soon see that the function either does not have any primitive function or has infinitely many.

**Theorem 2** (Set of primitive functions)**.** *Let $F$ be a primitive function of $f$ on $I$. Then the set of all primitive functions of $f$ on $I$ is*

$$\{F + c \mid c \in \mathbb{R}\} .$$

*Therefore, all primitive functions of $f$ are obtained by shifting any primitive function of $f$ by a constant.*

*Proof.* The derivative of a constant function is zero, so $(F + c)' = F' + 0 = f$ for every $c \in \mathbb{R}$ and every primitive function $F$ of $f$ on $I$. On the other hand, if $F$ and $G$ are primitive functions of $f$ on $I$ , then their difference $H = G - F$ has a zero derivative on $I$: for each $\gamma \in I$, we have $H'(\gamma) = G'(\gamma) - F'(\gamma) = f(\gamma) - f(\gamma) = 0$. Thus, for any two points $\alpha < \beta$ from $I$, according to Lagrange mean value theorem, we have

$$H(\beta) - H(\alpha) = (\beta - \alpha)H'(\gamma) = (\beta - \alpha)0 = 0$$

for some $\gamma \in (\alpha, \beta)$, so $H(\alpha) = H(\beta)$, so $H$ is constant on $I$. Thus, there is a constant $c$, such that that $G(x) - F(x) = c$ for every $x \in I$ and $G = F + c$. □

**Notation.** The fact that the function $F$ is a primitive function of $f$ is denoted by

$$\int f = F + c, \ c \in \mathbb{R} ,$$

to emphasize that $F$ shifted by a constant is also a primitive function of $f$. The symbol $\int f$ is to be understood as the set of all primitive functions of $f$ on the given interval.

# Primitive function and continuity

**Theorem 3** (Continuity of a primitive function)**.** *If $F$ is a primitive function of $f$ on $I$, then $F$ is continuous on $I$.*

*Proof.* We know from the winter term that the existence of the proper derivative of a function at a point implies its continuity at the given point. Since $F'(\alpha)$ exists and is equal to $f(\alpha)$ for each $\alpha \in I$, $F$ is continuous on $I$. □

**Theorem 4** (Continuous function has a primitive function)**.** *If $f$ is continuous on $I$, then $f$ has a primitive function $F$ on $I$.*

*Proof.* Later. $\square$

Can a discontinuous function have a primitive function? Yes.

**Example 1** (Discontinuous function with primitive function.)**.** *The function $f:\ \mathbb{R} \to \mathbb{R}$ defined as*

$$f(x) = 2\sin\left(\frac{1}{x^2}\right) - \frac{2\cos\left(\frac{1}{x^2}\right)}{x} \text{ for } x \neq 0, f(0) = 0\ ,$$

*has a primitive function on $\mathbb{R}$, even though it is not continuous at* 0.

*Proof.* Consider $F:\ \mathbb{R} \to \mathbb{R}$ defined for $x \neq 0$ as $F(x) = x^2 \sin(x^{-2})$ and for $x = 0$ as $F(0) = 0$. For $x \neq 0$ we have $F' = f$ by standard calculations. At zero by the definition of the derivative we calculate that

$$F'(0) = \lim_{x\to 0} \frac{F(x) - F(0)}{x - 0} = \lim_{x\to 0} x\sin(x^{-2}) = 0\ ,$$

because $|x\sin(x^{-2})| \leq |x|$ for every $x \neq 0$. Thus $F'(0)$ exists and again $F'(0) = f(0)$. Therefore, $F' = f$ on $\mathbb{R}$ and $F$ is a primitive function of $f$ on $\mathbb{R}$. Function $f$ is not continuous in 0, in every neighborhood of zero it is even unbounded from both above and below — for $x \to 0$ the graph oscilates with increasing amplitude and frequency. $\square$

In winter term, it was shown that the continuous function on the interval attains all values between its minium and maximum on the interval, so, its image is an interval. This property of a function is called *Darboux property*, according to the French mathematician Jean-Gaston Darboux (1842–1917). Darboux proved that functions with a primitive function have this property.

**Theorem 5** (Function with a primitive function has Darboux property)**.** *If $f$ has a primitive function $F$ on $I$, then $f$ has Darboux property on $I$.*

*Proof.* Let $x_1, x_2$ be any two points of $I$ such that $x_1 < x_2$, assume $f(x_1) < f(x_2)$ and consider $c \in \mathbb{R}$ satisfying $f(x_1) < c < f(x_2)$. (If $f(x_1) > c > f(x_2)$, the following argument is easily adjusted by replacing the minimum to maximum.) We find $x^* \in I$, in particular $x^* \in (x_1, x_2)$ such that $f(x^*) = c$. The function

$$H(x) = F(x) - cx$$

is continuous on $I$ (since $F$ is continuous by Theorem 3), moreover it has a proper derivative on $I$.

$$H'(x) = (F(x) - cx)' = f(x) - c\ .$$

According to the theorem from the winter term, $H$ attains minimum at some point $x^*$ on a compact interval $[x_1, x_2]$. Since $H'(x_1) = f(x_1) - c < 0$, thus, for some $\delta > 0$ we have $x \in (x_1, x_1 + \delta) \Rightarrow H(x) < H(x_1)$. Therefore, $x^* \neq x_1$. Similarly from $H'(x_2) > 0$ it follows that $x^* \neq x_2$. Thus $x^* \in (x_1, x_2)$ and according to the extreme criterion from the winter term, we must have $H'(x^*) = f(x^*) - c = 0$. So $f(x^*) = c$. □

**Consequence (an example of a function without a primitive function).** *Function* $\operatorname{sgn} : \mathbb{R} \to \mathbb{R}$*, defined as* $\operatorname{sgn}(x) = -1$ *for* $x < 0$*,* $\operatorname{sgn}(0) = 0$ *and* $\operatorname{sgn}(x) = 1$ *for* $x > 0$ *has no primitive function on* $\mathbb{R}$ *(or any other interval containing* $0$*).*

*Proof.* The function sgn does not have Darboux property: it attains values $-1$ and 1, but not $\frac{1}{2} \in (-1, 1)$. □

# Primitive functions of elementary functions

By reversing the direction of formulas for derivatives of elementary functions we get the following table of primitive functions (additive constant $c$ is omitted).

**Task.** *Taking derivative we have* $(\log x)' = 1/x$*, but also* $(\log(-x))' = (1/x)(-1) = 1/x$*. But* $\log x$ *and* $\log(-x)$ *do not differ just by shifting the constant, so the* $1/x$ *function has two fundamentally different primitive functions, contrary to the statement. How is it possible?*

| function | primitive function | on interval |
|---|---|---|
| $x^\alpha,\ \alpha \in \mathbb{R}\backslash\{-1\}$ | $\dfrac{x^{\alpha+1}}{\alpha+1}$ | $(0, +\infty)$ |
| $x^\alpha,\ \alpha \in \mathbb{Z},\ \alpha < -1$ | $\dfrac{x^{\alpha+1}}{\alpha+1}$ | $(0, +\infty)$ and $(-\infty, 0)$ |
| $x^\alpha,\ \alpha \in \mathbb{Z},\ \alpha > -1$ | $\dfrac{x^{\alpha+1}}{\alpha+1}$ | $\mathbb{R}$ |
| $x^{-1} = \dfrac{1}{x}$ | $\log \lvert x \rvert$ | $(0, +\infty)$ and $(-\infty, 0)$ |
| $\exp x = \mathrm{e}^x$ | $\exp x = \mathrm{e}^x$ | $\mathbb{R}$ |
| $\sin x$ | $-\cos x$ | $\mathbb{R}$ |
| $\cos x$ | $\sin x$ | $\mathbb{R}$ |
| $\dfrac{1}{\cos^2 x}$ | $\tan x = \dfrac{\sin x}{\cos x}$ | $((k - \frac{1}{2})\pi, (k + \frac{1}{2})\pi),\ k \in \mathbb{Z}$ |
| $\dfrac{1}{\sin^2 x}$ | $-\cot x = -\dfrac{\cos x}{\sin x}$ | $(k\pi, (k+1)\pi),\ k \in \mathbb{Z}$ |
| $\dfrac{1}{1+x^2}$ | $\arctan x$ | $\mathbb{R}$ |
| $\dfrac{1}{\sqrt{1-x^2}}$ | $\arcsin x$ | $(-1, 1)$ |

The table does not include hyperbolic functions (eg $\sinh x = \frac{\exp x - \exp(-x)}{2}$) goniometric functions (e.g., sekans $\sec x = \frac{1}{\cos x}$, popular in the US).