Lecture 1 (20.2.2019)

(translated and slightly adapted from lecture notes by Martin Klazar)
(Warning: not a substitute for attending the lectures, probably contains typos. Let me know if you spot any!)

Primitive functions

Definition 1 (Primitive function). *If $I \subseteq \mathbb{R}$ is a non-empty open interval and*

$$F, f:\ I \to \mathbb{R}$$

are functions satisfying $F' = f$ on I, we call F a primitive function of *f on I.*

First, some motivation — the relation of primitive functions with planar figures. For a nonnegative and continuous function $f:\ [a,b] \to \mathbb{R}$ we consider a planar figure

$$U(a,b,f) = \{(x,y) \in \mathbb{R}^2 \mid a \le x \le b \ \&\ 0 \le y \le f(x)\}\ .$$

Its area, whatever it is, is denoted by

$$\int_a^b f := \mathrm{area}(U(a,b,f))\ .$$

This is the area of part of the plane defined by the axis x, graph of the function f and the vertical lines $y = a$ and $y = b$. Two basic relationships between the area and the derivative are as follows. Consider the function F of x defined as the area $U(a,x,f)$, i.e., $F(x) = \int_a^x f$. *The first fundamental theorem of calculus* says that for every $c \in [a,b]$ we have

$$F'(c) = f(c)$$

— the derivative of a function whose argument x is the upper limit of the $U(a,x,f)$ and the value is its area is equal to the original function f. Thus, the function $F(x) = \int_a^x f$ is a primitive function of f. According to the *second fundamental theorem of calculus* for every function F, which is a primitive function of f on $[a,b]$, it holds that

$$\int_a^b f = F(b) - F(a)\ .$$

If we know a primitive function of f (many can be deduced by simply reversing the rules for derivative of elementary functions), we can immediately calculate the area of $U(a,b,f)$. We formulate and prove both theorems precisely later in the lecture on Riemann's integral, when we also introduce the area $\int_a^b f$. But first we need to look at the properties of primitive functions — where the function has a primitive function, whether it is unique, etc.

Due to the linearity of the derivative, primitive functions are also linear:

Theorem 1 (Linearity of primitive functions)**.** *If F is a primitive function of f, and G is a primitive function of g on an interval I and $\alpha, \beta \in \mathbb{R}$, then the function*

$$\alpha F + \beta G$$

is a primitive function of $\alpha f + \beta g$ on I.

For limit and derivative, the result of the operation is unique if it exists, but a primitive function is not unique. We will soon see that the function either does not have any primitive function or has infinitely many.

Theorem 2 (Set of primitive functions)**.** *Let F be a primitive function of f on I. Then the set of all primitive functions of f on I is*

$$\{F + c \mid c \in \mathbb{R}\} .$$

Therefore, all primitive functions of f are obtained by shifting any primitive function of f by a constant.

Proof. The derivative of a constant function is zero, so $(F + c)' = F' + 0 = f$ for every $c \in \mathbb{R}$ and every primitive function F of f on I. On the other hand, if F and G are primitive functions of f on I , then their difference $H = G - F$ has a zero derivative on I: for each $\gamma \in I$, we have $H'(\gamma) = G'(\gamma) - F'(\gamma) = f(\gamma) - f(\gamma) = 0$. Thus, for any two points $\alpha < \beta$ from I, according to Lagrange mean value theorem, we have

$$H(\beta) - H(\alpha) = (\beta - \alpha)H'(\gamma) = (\beta - \alpha)0 = 0$$

for some $\gamma \in (\alpha, \beta)$, so $H(\alpha) = H(\beta)$, so H is constant on I. Thus, there is a constant c, such that that $G(x) - F(x) = c$ for every $x \in I$ and $G = F + c$. □

Notation. The fact that the function F is a primitive function of f is denoted by

$$\int f = F + c, \ c \in \mathbb{R} ,$$

to emphasize that F shifted by a constant is also a primitive function of f. The symbol $\int f$ is to be understood as the set of all primitive functions of f on the given interval.

Primitive function and continuity

Theorem 3 (Continuity of a primitive function)**.** *If F is a primitive function of f on I, then F is continuous on I.*

Proof. We know from the winter term that the existence of the proper derivative of a function at a point implies its continuity at the given point. Since $F'(\alpha)$ exists and is equal to $f(\alpha)$ for each $\alpha \in I$, F is continuous on I. □

Theorem 4 (Continuous function has a primitive function)**.** *If f is continuous on I, then f has a primitive function F on I.*

Proof. Later. □

Can a discontinuous function have a primitive function? Yes.

Example 1 (Discontinuous function with primitive function.)**.** *The function $f:\ \mathbb{R} \to \mathbb{R}$ defined as*

$$f(x) = 2\sin\left(\frac{1}{x^2}\right) - \frac{2\cos\left(\frac{1}{x^2}\right)}{x} \quad \textit{for } x \neq 0, f(0) = 0\ ,$$

has a primitive function on $\mathbb{R}$, even though it is not continuous at 0.

Proof. Consider $F:\ \mathbb{R} \to \mathbb{R}$ defined for $x \neq 0$ as $F(x) = x^2 \sin(x^{-2})$ and for $x = 0$ as $F(0) = 0$. For $x \neq 0$ we have $F' = f$ by standard calculations. At zero by the definition of the derivative we calculate that

$$F'(0) = \lim_{x\to 0} \frac{F(x) - F(0)}{x - 0} = \lim_{x\to 0} x\sin(x^{-2}) = 0\ ,$$

because $|x\sin(x^{-2})| \leq |x|$ for every $x \neq 0$. Thus $F'(0)$ exists and again $F'(0) = f(0)$. Therefore, $F' = f$ on $\mathbb{R}$ and F is a primitive function of f on $\mathbb{R}$. Function f is not continuous in 0, in every neighborhood of zero it is even unbounded from both above and below — for $x \to 0$ the graph oscilates with increasing amplitude and frequency. □

In winter term, it was shown that the continuous function on the interval attains all values between its minium and maximum on the interval, so, its image is an interval. This property of a function is called *Darboux property*, according to the French mathematician Jean-Gaston Darboux (1842–1917). Darboux proved that functions with a primitive function have this property.

Theorem 5 (Function with a primitive function has Darboux property)**.** *If f has a primitive function F on I, then f has Darboux property on I.*

Proof. Let x_1, x_2 be any two points of I such that $x_1 < x_2$, assume $f(x_1) < f(x_2)$ and consider $c \in \mathbb{R}$ satisfying $f(x_1) < c < f(x_2)$. (If $f(x_1) > c > f(x_2)$, the following argument is easily adjusted by replacing the minimum to maximum.) We find $x^* \in I$, in particular $x^* \in (x_1, x_2)$ such that $f(x^*) = c$. The function

$$H(x) = F(x) - cx$$

is continuous on I (since F is continuous by Theorem 3), moreover it has a proper derivative on I.

$$H'(x) = (F(x) - cx)' = f(x) - c\ .$$

According to the theorem from the winter term, H attains minimum at some point x^* on a compact interval $[x_1, x_2]$. Since $H'(x_1) = f(x_1) - c < 0$, thus, for some $\delta > 0$ we have $x \in (x_1, x_1 + \delta) \Rightarrow H(x) < H(x_1)$. Therefore, $x^* \neq x_1$. Similarly from $H'(x_2) > 0$ it follows that $x^* \neq x_2$. Thus $x^* \in (x_1, x_2)$ and according to the extreme criterion from the winter term, we must have $H'(x^*) = f(x^*) - c = 0$. So $f(x^*) = c$. □

Consequence (an example of a function without a primitive function). *Function* $\mathrm{sgn} : \mathbb{R} \to \mathbb{R}$*, defined as* $\mathrm{sgn}(x) = -1$ *for* $x < 0$*,* $\mathrm{sgn}(0) = 0$ *and* $\mathrm{sgn}(x) = 1$ *for* $x > 0$ *has no primitive function on* $\mathbb{R}$ *(or any other interval containing* 0*).*

Proof. The function sgn does not have Darboux property: it attains values -1 and 1, but not $\frac{1}{2} \in (-1, 1)$. □

Primitive functions of elementary functions

By reversing the direction of formulas for derivatives of elementary functions we get the following table of primitive functions (additive constant c is omitted).

Task. *Taking derivative we have* $(\log x)' = 1/x$*, but also* $(\log(-x))' = (1/x)(-1) = 1/x$*. But* $\log x$ *and* $\log(-x)$ *do not differ just by shifting the constant, so the* $1/x$ *function has two fundamentally different primitive functions, contrary to the statement. How is it possible?*

function	primitive function	on interval
$x^\alpha,\ \alpha \in \mathbb{R}\setminus\{-1\}$	$\dfrac{x^{\alpha+1}}{\alpha+1}$	$(0,+\infty)$
$x^\alpha,\ \alpha \in \mathbb{Z},\ \alpha < -1$	$\dfrac{x^{\alpha+1}}{\alpha+1}$	$(0,+\infty)$ and $(-\infty,0)$
$x^\alpha,\ \alpha \in \mathbb{Z},\ \alpha > -1$	$\dfrac{x^{\alpha+1}}{\alpha+1}$	$\mathbb{R}$
$x^{-1} = \dfrac{1}{x}$	$\log\lvert x\rvert$	$(0,+\infty)$ and $(-\infty,0)$
$\exp x = \mathrm{e}^x$	$\exp x = \mathrm{e}^x$	$\mathbb{R}$
$\sin x$	$-\cos x$	$\mathbb{R}$
$\cos x$	$\sin x$	$\mathbb{R}$
$\dfrac{1}{\cos^2 x}$	$\tan x = \dfrac{\sin x}{\cos x}$	$((k-\frac{1}{2})\pi,(k+\frac{1}{2})\pi),\ k \in \mathbb{Z}$
$\dfrac{1}{\sin^2 x}$	$-\cot x = -\dfrac{\cos x}{\sin x}$	$(k\pi,(k+1)\pi),\ k \in \mathbb{Z}$
$\dfrac{1}{1+x^2}$	$\arctan x$	$\mathbb{R}$
$\dfrac{1}{\sqrt{1-x^2}}$	$\arcsin x$	$(-1,1)$

The table does not include hyperbolic functions (eg $\sinh x = \frac{\exp x - \exp(-x)}{2}$) goniometric functions (e.g., sekans $\sec x = \frac{1}{\cos x}$, popular in the US).

Lecture 2 (27.2.2019)

(translated and slightly adapted from lecture notes by Martin Klazar)
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Methods for computing primitive functions

To calculate the derivative of the product of two functions, we have the Leibniz formula $(fg)' = f'g + fg'$. By inverting it, we will get the following important result for primitive functions.

Theorem 6 (Integration per partes (by parts)). *If $f, g : I \to \mathbb{R}$ are continuous functions on an interval I and F, G their corresponding primitive functions on I then the following equality holds on I:*

$$\int fG + \int Fg = FG + c\ .$$

In other words, the functions fG and Fg have primitive functions on I whose sum is always equal to function FG on I, up to the additive constant c.

Proof. Since that f and g are continuous on I, and by Theorem 3, the primitive functions F and G are also continuous. So products fG and Fg are also continuous, and by Theorem 4, they have primitive functions $\int fG$ and $\int Fg$ on I. By linearity of primitive functions, the sum $\int fG + \int Fg$ is a primitive function of $fG + Fg$. Moreover, the FG function is a primitive function of $fG + Fg$, because the Leibniz formula gives $(FG)' = fG + Fg$. Thus, we get that $\int fG + \int Fg = FG + c$. $\square$

The formula for integration per partes is usually given in an equivalent form

$$\int F'G = FG - \int FG'$$

So if we can calculate the primitive function of FG' for the two functions F and G with continuous derivatives ($F' = f$ and $G' = g$) we get a primitive function of $F'G$ according to this formula.

Example 2. *With $x' = 1$ and $(\log x)' = 1/x$ on the interval $(0, +\infty)$ we have*

$$\int \log x = \int x' \log x = x \log x - \int x(\log x)' = x \log x - \int 1 = x \log x - x + c$$

on $(0, +\infty)$. By taking derivative, we can easily check the correctness of the derived formula.

Inverting the rule for derivative of the product gives the formula for integration per partes and by inverting the rule for derivative of the composed function we get a formula for integration by substitution. It has two forms, according to the direction of reading the equality of $f(\varphi)' = f'(\varphi)\varphi'$.

Theorem 7 (Integration by substitution)**.** *Let* $\varphi:\ (\alpha,\beta) \to (a,b)$ *and* $f:$ $(a,b) \to \mathbb{R}$ *be two functions such that* φ *has a proper derivative* φ' *on* (α,β)*.*

1. *If* $F = \int f$ *on* (a,b)*, then* $\int f(\varphi)\varphi' = F(\varphi) + c$ *on* (α,β)*.*
2. *Suppose* φ *additionally that* $\varphi((\alpha,\beta)) = (a,b)$ *and either* $\varphi' > 0$ *or* $\varphi' < 0$ *on* (α,β)*. If* $G = \int f(\varphi)\varphi'$ *on* (α,β)*, then* $\int f = G(\varphi^{\langle -1\rangle}$ *on* (a,b)*.*

Proof. The first part follows immediately by the derivative:

$$F(\varphi)' = F'(\varphi)\varphi' = f(\varphi)\varphi'$$

on (α,β), from the assumption about F and derivative of the composed function.

In the second part assumptions about φ guarantee that it is a strictly increasing or a strictly decreasing bijection from (α,β) to (a,b). So it is an injective function, it has an inverse function

$$\varphi^{\langle -1\rangle} : (a,b) \to (\alpha,\beta).$$

We can compute derivative of this function using the inverse function derivative rule. This gives, together with the assumption about G, derivative of the composed function and the derivative of the inverse function, that $G(\varphi^{\langle -1\rangle})$ is primitive function of f on (a,b) :

$$G(\varphi^{\langle -1\rangle})' = G'(\varphi^{\langle -1\rangle}) \cdot (\varphi^{\langle -1\rangle})' = f(\varphi(\varphi^{\langle -1\rangle}))\varphi'(\varphi^{\langle -1\rangle}) \cdot \frac{1}{\varphi'(\varphi^{\langle -1\rangle})} = f\ .$$

□

Here are two examples of both forms of the substitution rule.

1. When $F(x) = \int f(x)\,dx$ at some I a $a,b \in \mathbb{R}$, $a \neq 0$, then according to the first part we calculate that

$$\int f(ax+b)\,dx = a^{-1}\int f(ax+b)\cdot(ax+b)'\,dx = a^{-1}F(ax+b) + c\ ,$$

on the interval $J = a^{-1}(I-b) = \{a^{-1}(x-b) \mid x \in I\}$. It is easy to check backwards by taking derivative. We took $\varphi(x) = ax+b$.

2. We want to calculate the primitive function of $\sqrt{1-t^2}$ on $(-1,1)$. Because it resembles the derivative of arcsin, we try the substitution $t = \varphi(x) = \sin x:\ (-\frac{\pi}{2},\frac{\pi}{2}) \to (-1,1)$. The assumptions of the second form of the substitution rule are fulfilled.

$$G(x) = \int \sqrt{1-\sin^2 x} \cdot (\sin x)' \, dx = \int \sqrt{\cos^2 x} \cdot \cos x \, dx = \int \cos^2 x \, dx \; .$$

Did it help? It helped because the last primitive function can easily be calculated by integrating per partes:

$$\begin{aligned}\int \cos^2 x &= \int \cos x (\sin x)' = \cos x \sin x + \int \sin^2 x \\ &= \cos x \sin x + \int (1-\cos^2 x) \\ &= \sin x \cos x + x - \int \cos^2 x \; ,\end{aligned}$$

so,

$$G(x) = \int \cos^2 x = \frac{\sin x \cos x + x}{2} + c = \frac{\sin x \sqrt{1-\sin^2 x} + x}{2} + c \; .$$

After letting $x = \varphi^{\langle -1 \rangle}(t) = \arcsin t$ we get the desired result

$$\int \sqrt{1-t^2} = G(\arcsin t) + c = \frac{t\sqrt{1-t^2} + \arcsin t}{2} + c, \quad \text{on } (-1,1) \; .$$

By derivative, we can easily verify it is correct.

By saying that *f can be expressed using elementary functions* we mean that f can be expressed from the basic functions $\exp(x)$ (exponential), $\log x$, $\sin x$, $\arcsin x$, $\cos x$, $\arccos x$, $\tan x$ and $\arctan x$ repeatedly using the arithmetic operations $+, -, \times, :$, and the folding operations. Many primitive functions can be expressed in this way, but many primitive functions cannot. The following theorem, which we will not prove, gives some important examples of such functions.

Theorem 8 (Non-elementary primitive functions)**.** *Primitive functions*

$$F_1(x) = \int \exp(x^2), \;\; F_2(x) = \int \frac{\sin x}{x} \quad \textit{and} \quad F_3(x) = \int \frac{1}{\log x}$$

(on the intervals where they are defined) cannot be expressed using elementary functions.

Primitive functions of rational functions

A relatively wide class of functions to which primitive functions can be computed are *rational functions*, which are fractions of polynomials. Let's give a simple example. Let $I \subset \mathbb{R}$ be any open interval that does not contain -1 and 1. Then

$$\begin{aligned}
\int \frac{x^2}{x^2-1} &= \int \Big(1 + \frac{1}{x^2-1}\Big) = \int \Big(1 + \frac{1/2}{x-1} - \frac{1/2}{x+1}\Big) \\
&= \int 1 + \frac{1}{2}\int \frac{1}{x-1} - \frac{1}{2}\int \frac{1}{x+1} \\
&= x + \frac{\log|x-1| - \log|x+1|}{2} + c \\
&= x + \log(\sqrt{|(x-1)/(x+1)|}) + c
\end{aligned}$$

on I . It turns out that similarly, a primitive function can be calculated for any rational function. The key is a decomposition to the sum simpler rational functions (the first line of calculation), which is called decomposition into *partial fractions*. In the following we use some results from algebra that we will not prove here.

Theorem 9 (Primitive function for rational function can always be calculated)**.** *Let $P(x)$ and $Q(x) \neq 0$ be polynomials with real coefficients and $I \subset \mathbb{R}$ is an open interval not containing no roots of $Q(x)$. Primitive function*

$$F(x) = \int \frac{P(x)}{Q(x)} \qquad \textit{(on } I\textit{)}$$

can be expressed using elementary functions, namely using rational functions, logarithms and arcustangent.

Proof. Without loss of generality, assume that $Q(x)$ is monic (i.e. its leading coefficient is 1). After dividing $P(x)$ by $Q(x)$ with remainder we have

$$\frac{P(x)}{Q(x)} = p(x) + \frac{R(x)}{Q(x)} \ ,$$

where $p(x), R(x)$ are real polynomials and $R(x)$ has smaller degree than $Q(x)$. There is unique way to express $Q(x)$ as a product of irreducible real polynomials (i.e. polynomials that cannot be expressed as product of polynomials of smaller degree), moreover, these polynomials will have degree at most 2:

$$Q(x) = \prod_{i=1}^{k}(x-\alpha_i)^{m_i} \prod_{i=1}^{l}(x^2+\beta_i x+\gamma_i)^{n_i} \ ,$$

where $k, l \geq 0$ are integers, $\alpha_i, \beta_i, \gamma_i \in \mathbb{R}$, $m_i, n_i \geq 1$ are integers, numbers α_i are pairwise distinct, pairs (β_i, γ_i) are pairwise distinct and always $\beta_i^2 - 4\gamma_i < 0$ (thus, the polynomial $x^2+\beta_i x+\gamma_i$ is irreducible as it has no real roots). It can be shown that $R(x)/Q(x)$ has unique expression as the sum of *partial fractions*

$$\frac{R(x)}{Q(x)} = \sum_{i=1}^{k}\sum_{j=1}^{m_i} \frac{\delta_{i,j}}{(x-\alpha_i)^j} + \sum_{i=1}^{l}\sum_{j=1}^{n_i} \frac{\epsilon_{i,j}x+\theta_{i,j}}{(x^2+\beta_i x+\gamma_i)^j} \ ,$$

where $\delta_{i,j}, \epsilon_{i,j}, \theta_{i,j} \in \mathbb{R}$. In the previous example we have $P(x) = x^2$, $Q(x) = x^2 - 1$, $p(x) = 1$, $R(x) = 1$, $k = 2$, $m_1 = m_2 = 1$, $l = 0$ (decomposition of $Q(x)$ contains no quadratic polynomial with three non-zero coefficients), $\alpha_1 = 1$, $\alpha_2 = -1$, $\delta_{1,1} = \frac{1}{2}$ a $\delta_{2,1} = -\frac{1}{2}$. Thus, a primitive function $\int \frac{P(x)}{Q(x)}$ equals to the sum of finitely many primitive functions of three types:

$$\int p(x), \int \frac{\delta}{(x-\alpha)^j} \quad \text{a} \quad \int \frac{\epsilon x + \theta}{(x^2 + \beta x + \gamma)^j} ,$$

where $p(x)$ is a real polynomial, $j \in \mathbb{N}$ and except x all other symbols are real constants, and $\beta^2 - 4\gamma < 0$. If we can express these primitive functions using elementary functions, we can express $\int \frac{P(x)}{Q(x)}$ using elementary functions as well.

It is easy to calculate primitive functions of the first two types:

$$\int p(x) = \int (a_n x^n + \ldots + a_1 x + a_0) = \frac{a_n x^{n+1}}{n+1} + \ldots + \frac{a_1 x^2}{2} + a_0 x$$

on $\mathbb{R}$ and

$$\int \frac{\delta}{(x-\alpha)^j} = \frac{\delta}{(1-j)(x-\alpha)^{j-1}} \ (j \geq 2), \quad \int \frac{\delta}{x-\alpha} = \delta \log |x - \alpha|$$

on $(-\infty, \alpha)$ and $(\alpha, +\infty)$ (we omitted additive constants). The third type is more complex. We have

$$\int \frac{\epsilon x + \theta}{(x^2 + \beta x + \gamma)^j} = \frac{\epsilon}{2} \int \frac{2x + \beta}{(x^2 + \beta x + \gamma)^j} + (\theta - \epsilon\beta/2) \int \frac{1}{(x^2 + \beta x + \gamma)^j} .$$

For the last but one $\int$ is after substituting $y = x^2 + \beta x + \gamma$ of the second typewe have

$$\int \frac{2x + \beta}{(x^2 + \beta x + \gamma)^j} = \frac{1}{(1-j)(x^2 + \beta x + \gamma)^{j-1}} \ (j \geq 2)$$

and

$$\int \frac{2x + \beta}{x^2 + \beta x + \gamma} = \log |x^2 + \beta x + \gamma| = \log(x^2 + \beta x + \gamma) .$$

on $\mathbb{R}$ (recall that $x^2 + \beta x + \gamma$ has no real root) . It remains to calculate a primitive function $\int 1/(x^2 + \beta x + \gamma)^j$. We denote $\eta = \sqrt{\gamma - \beta^2/4}$ (recall that $\gamma - \beta^2/4 > 0$) and use substitution $y = y(x) = x/\eta + \beta/2\eta$. By completing the square we get

$$\begin{aligned}
\int \frac{1}{(x^2 + \beta x + \gamma)^j} &= \frac{1}{\eta^{2j-1}} \int \frac{1/\eta}{((x/\eta + \beta/2\eta)^2 + 1)^j} \\
&= \frac{1}{\eta^{2j-1}} \int \frac{y'}{((x/\eta + \beta/2\eta)^2 + 1)^j} \\
&= \frac{1}{\eta^{2j-1}} \int \frac{1}{(y^2 + 1)^j} .
\end{aligned}$$

Thus, it remains to compute the following primitive function on $\mathbb{R}$:

$$I_j = \int \frac{1}{(1+x^2)^j} \ .$$

For $j = 1$ we already know that $I_1 = \arctan x$. For $j = 2, 3, \ldots$ we express I_j using recurrence obtained by integration by parts:

$$\begin{aligned} I_j &= \int \frac{x'}{(1+x^2)^j} = \frac{x}{(1+x^2)^j} + \int \frac{2jx^2}{(1+x^2)^{j+1}} \\ &= \frac{x}{(1+x^2)^j} + 2j \int \frac{x^2+1}{(1+x^2)^{j+1}} - 2j \int \frac{1}{(1+x^2)^{j+1}} \\ &= \frac{x}{(1+x^2)^j} + 2jI_j - 2jI_{j+1} \ , \end{aligned}$$

thus

$$I_{j+1} = I_j(1 - 1/2j) + \frac{x}{2j(1+x^2)^j} \ .$$

For instance,

$$I_2 = \frac{\arctan x}{2} + \frac{x}{2(1+x^2)} \ \text{ a } \ I_3 = \frac{3\arctan x}{8} + \frac{3x}{8(1+x^2)} + \frac{x}{4(1+x^2)^2} \ .$$

In general, the recurrence shows that for every $j = 1, 2, \ldots$, I_j has form $I_j = \kappa \arctan x + r(x)$, where κ is a fraction and $r(x)$ is a rational function. Thus, we have completed the calculation of the primitive function of the third type from the expression $R(x)/Q(x)$ of the sum of the partial fractions and obtained a complete expression of the primitive function $\int \frac{P(x)}{Q(x)}$ using elementary functions. $\square$

Lecture 3 (6.3.2019)

(translated and slightly adapted from lecture notes by Martin Klazar)

Riemann integral

Now we define precisely the concept of the area, in particular, the area of figure $U(a, b, f)$ under the graph of a function f. Let $-\infty < a < b < +\infty$ be two real numbers and $f : [a, b] \to \mathbb{R}$ any function that may not be continuous or bounded. The finite $k+1$-tuple of points $D = (a_0, a_1, \dots, a_k)$ from the interval $[a, b]$ is called a *partition* of $[a, b]$ if

$$a = a_0 < a_1 < a_2 < \ldots < a_k = b \; .$$

These points divide the interval $[a, b]$ into intervals $I_i = [a_{i-1}, a_i]$. We denote by $|I_i|$ the length of interval I_i: $|I_i| = a_i - a_{i-1}$ and $|[a, b]| = b - a$. Clearly

$$\sum_{i=1}^{k} |I_i| = (a_1 - a_0) + (a_2 - a_1) + \ldots + (a_k - a_{k-1}) = b - a = |[a, b]| \; .$$

Norm of a partition D is the maximum length of an interval of the partition and is denoted by λ:

$$\lambda = \lambda(D) = \max_{1 \le i \le k} |I_i| \; .$$

Partition of an interval $[a, b]$ *with points* is a pair (D, C) where $D = (a_0, a_1, \dots, a_k)$ is a partition of $[a, b]$ and a k-tuple $C = (c_1, c_2, \dots, c_k)$ consists of $c_i \in I_i$ (i.e. $a_{i-1} \le c_i \le a_i$). *Riemann sum* corresponding to the function f and a partition with points (D, C) is defined as

$$R(f, D, C) = \sum_{i=1}^{k} |I_i| f(c_i) = \sum_{i=1}^{k} (a_i - a_{i-1}) f(c_i) \; .$$

If $f \ge 0$ on $[a, b]$, it is the sum of k rectangles $I_i \times [0, f(c_i)]$ whose union approximates figure $U(a, b, f)$. However, Riemann sum is defined for every function f, regardless of its sign on $[a, b]$. The following definition was introduced by Bernhard Riemann (1826–1866).

Definition 2 (First definition of Riemann integral, Riemann)**.** *We say that* $f : [a, b] \to \mathbb{R}$ *has Riemann integral* $I \in \mathbb{R}$ *on* $[a, b]$ *if for every* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that for each partition of* $[a, b]$ *with points* (D, C) *such that* $\lambda(D) < \delta$ *the following holds:*

$$|I - R(f, D, C)| < \varepsilon$$

We require $I \in \mathbb{R}$, values $\pm\infty$ are not allowed (although, it is possible to define them). If there is such a number I, we write

$$I = \int_a^b f(x)\, dx = \int_a^b f = (\mathcal{R}) \int_a^b f$$

and say that f is *Riemann integrable* on the interval $[a, b]$. We will work with the class of all Riemann integrable functions

$$\mathcal{R}(a, b) := \{f \mid f \text{ is defined and Riemann integrable on } [a, b]\}.$$

Thus, the first definition of the Riemann integral can be summarized by the formula

$$\int_a^b f = \lim_{\lambda(D)\to 0} R(f, D, C) \in \mathbb{R} .$$

We understand the limit here as defined in the definition above; as a symbol, we defined only limit of a sequence and of a function in a point.

For the second, equivalent, definition of the integral we will need a few more concepts. For $f :\ [a, b] \to \mathbb{R}$ and a partition $D = (a_0, a_1, \ldots, a_k)$ of interval $[a, b]$ we define *lower and upper Riemann sum*, respectively, (even though they were introduced by Darboux) as

$$s(f, D) = \sum_{i=1}^{k} |I_i| m_i, \text{ and } S(f, D) = \sum_{i=1}^{k} |I_i| M_i ,$$

where

$$m_i = \inf_{x\in I_i} f(x) \text{ and } M_i = \sup_{x\in I_i} f(x)$$

$$I_i = [a_{i-1}, a_i]$$

These sums are always defined $s(f, D) \in \mathbb{R} \cup \{-\infty\}$ and $S(f, D) \in \mathbb{R} \cup \{+\infty\}$ *Lower and upper Riemann integral*, respectively, of a function f on the interval $[a, b]$ is defined as

$$\underline{\int_a^b} f = \underline{\int_a^b} f(x)\, dx = \sup(\{s(f, D) :\ D \text{ is a partition of } [a, b]\}) ,$$

and

$$\overline{\int_a^b} f = \overline{\int_a^b} f(x)\, dx = \inf(\{S(f, D) :\ D \text{ is a partition of } [a, b]\}) .$$

These terms are always defined and we have $\underline{\int_a^b} f, \overline{\int_a^b} f \in \mathbb{R}^* = \mathbb{R} \cup \{-\infty, +\infty\}$.

Definition 3 (Second definition of Riemann integral, Darboux). *We say that* $f :\ [a, b] \to \mathbb{R}$ *has on* $[a, b]$ *Riemann integral, if*

$$\underline{\int_a^b} f(x)\, dx = \overline{\int_a^b} f(x)\, dx \in \mathbb{R} .$$

This common value, if it exists, is denoted by

$$\int_a^b f(x)\, dx = \int_a^b f$$

and we call it the Riemann integral of f *on the interval* $[a, b]$.

The two definitions are equivalent: they give the same classes of Riemann integrable functions and the same value of the Riemann integral, if defined.

Example 3 (Bounded function without integral)**.** *A function* $f: [0,1] \to \{0,1\}$ *defined as* $f(\alpha) = 1$ *when* α *is a rational number, and* $f(\alpha) = 0$*, when* α *is irrational, is called* Dirichlet function*, and does not have Riemann integral on* $[0,1]$*, although bounded.*

Each positive-length interval contains points where f *has a value of* 0*, as well as points that have a value of* 1*. Then* $s(f,D) = 0$ *and* $S(f,D) = 1$ *for every partition of* D *and therefore*

$$\underline{\int_0^1} f = 0 < \overline{\int_0^1} f = 1 \, .$$

Theorem 10 (Unbounded functions have no integral)**.** *If the* $f: [a,b] \to \mathbb{R}$ *function is not bounded then it does not have a Riemann integral on* $[a,b]$*, according to both definitions.*

When $D = (a_0, a_1, \ldots, a_k)$ a $D' = (b_0, b_1, \ldots, b_l)$ are partitions of $[a,b]$ and $D \subset D'$, that is for every $i = 0, 1, \ldots, k$ there exists j, such that $a_i = b_j$ (therefore $k \le l$), we say that D' *is a refinement of* D or that D' *refines* D.

Lemma 11 (Riemann sums of a refinement)**.** *If* $f: [a,b] \to \mathbb{R}$ *and* D, D' *are two partitions of* $[a,b]$*, and* D' *refines* D*,*

$$s(f,D') \ge s(f,D) \text{ and } S(f,D') \le S(f,D).$$

Proof. Considering the definition of $s(f,D)$ a $S(f,D)$ and the fact that D' can be created from D by adding points, it is enough to prove both inequalities in a situation where $D = (a_0 = a < a_1 = b)$ a $D' = (a'_0 = a < a'_1 < a'_2 = b)$. According to the definition of infima f, we have

$$m_0 = \inf_{a_0 \le x \le a_1} f(x) \le \inf_{a'_0 \le x \le a'_1} f(x) = m_0 \inf_{a'_1 \le x \le a'_2} f(x) = m'_1$$

Then

$$\begin{aligned} s(f,D') &= (a'_1 - a'_0)m'_0 + (a'_2 - a'_1)m'_1 \\ &\ge (a'_1 - a'_0)m_0 + (a'_2 - a'_1)m_0 \\ &= (a'_2 - a'_0)m_0 = (b-a)m_0 \\ &= s(f,D) \, . \end{aligned}$$

Proof of the inequality $S(f,D') \le S(f,D)$ is similar. □

Corollary 12. *When* $f: [a,b] \to \mathbb{R}$ *and* D, D' *are two partitions* $[a,b]$*, then*

$$s(f,D) \le S(f,D') \, .$$

Proof. Let $E = D \cup D'$ be a common refinement of both partitions. According to the previous lemma we have

$$s(f, D) \leq s(f, E) \leq S(f, E) \leq S(f, D')$$

More precisely, the first and last inequality follow from the previous lemma, and the middle one from the definition of upper and lower sum. $\square$

Theorem 13 (Lower integral does not exceed upper)**.** *Let* $f : [a, b] \to \mathbb{R}$, $m = \inf_{a \leq x \leq b} f(x)$, $M = \sup_{a \leq x \leq b} f(x)$ *and* D, D' *be two partitions of interval* $[a, b]$. *Then the following inequalities hold:*

$$m(b-a) \leq s(f, D) \leq \underline{\int_a^b} f \leq \overline{\int_a^b} f \leq S(f, D') \leq M(b-a) .$$

Proof. The first and last inequality are the special cases of the previous lemma. The second and penultimate inequality comes straight from the definition of the lower and upper integral as supremum or infimum respectively. According to the corollary, each element is a set of lower sums whose supremum is $\underline{\int_a^b} f$ smaller or equal to each element of the upper sum set whose infim is $\overline{\int_a^b} f$. Using the definition of infimum (the largest lower bound) and supremum (the smallest upper bound) we get the middle inequality: For each partition D, $s(f, D)$ the lower bound of the set of upper sums, that is, $s(f, D) \leq \overline{\int_a^b} f$, and so $\overline{\int_a^b} f$ is the upper bound of the set of lower sums, thus $\underline{\int_a^b} f \leq \overline{\int_a^b} f$. $\square$

Example 4. *We calculate by definition that*

$$\int_0^1 x \, dx = 1/2 .$$

For $n = 1, 2, \ldots$ *take a partition of* $D_n = (0, \frac{1}{n}, \frac{2}{n}, \ldots, 1)$. *Then*

$$s(f, D_n) = \sum_{i=1}^{n} \frac{1}{n}\Big(\frac{i-1}{n}\Big) = n^{-2}(0 + 1 + 2 + \ldots + (n-1))$$

similarly

$$S(f, D_n) = \sum_{i=1}^{n} \frac{1}{n}\Big(\frac{i}{n}\Big) = n^{-2}(1 + 2 + \ldots + n) .$$

Since $S(f, D_n) - s(f, D_n) = \frac{1}{n} \to 0$ *for* $n \to \infty$, $f(x) = x$ *has Riemann integral on* $[0, 1]$ *by Integrability criterion. Moreover, we have*

$$\underline{\int_0^1} x \, dx = \lim_{n \to \infty} s(f, D_n) = \lim_{n \to \infty} \frac{(n-1)n}{2} \cdot \frac{1}{n^2} = 1/2$$

$$\overline{\int_0^1} x \, dx = \lim_{n \to \infty} S(f, D_n) = \lim_{n \to \infty} \frac{n(n+1)}{2} \cdot \frac{1}{n^2} = 1/2$$

So, $\int_0^1 x \, dx = 1/2$.

Lecture 4 (13.3.2018)

(translated and slightly adapted from lecture notes by Martin Klazar)

Theorem 14 (Integrability criterion)**.** *Let $f:\ [a,b] \to \mathbb{R}$. Then*

$$f \in \mathcal{R}(a,b) \iff \forall \varepsilon > 0\ \exists D:\ 0 \le S(f,D) - s(f,D) < \varepsilon.$$

In other words, f has Riemann integral if and only if for every $\varepsilon > 0$ there exists a partition D of interval $[a,b]$ such that its upper Riemann sum is greater than the corresponding lower Riemann sum by less than ε.

Proof. " $\Rightarrow$ " We assume that f has R. integral on $[a,b]$, i.e., $\underline{\int_a^b} f = \overline{\int_a^b} f = \int_a^b f \in \mathbb{R}$. Let $\varepsilon > 0$ be given. By definition of the lower and upper integrals, there are partitions E_1 and E_2 so that

$$s(f,E_1) > \underline{\int_a^b} f - \frac{\varepsilon}{2} = \int_a^b f - \frac{\varepsilon}{2} \ \text{ a } \ S(f,E_2) < \overline{\int_a^b} f + \frac{\varepsilon}{2} = \int_a^b f + \frac{\varepsilon}{2}\ .$$

According to the lemma, these inequalities also apply after replacing E_1 and E_2 with their joint refinement $D = E_1 \cup E_2$. Summing up both inequalities we will get

$$S(f,D) - s(f,D) < \int_a^b f + \frac{\varepsilon}{2} + \Big(- \int_a^b f + \frac{\varepsilon}{2}\Big) = \varepsilon\ .$$

" $\Leftarrow$ " Given $\varepsilon > 0$ we take a partition of D satisfying the condition. According to the definition of the lower and upper integral we get

$$\overline{\int_a^b} f \le S(f,D) < s(f,D) + \varepsilon \le \underline{\int_a^b} f + \varepsilon, \text{ thus } \overline{\int_a^b} f - \underline{\int_a^b} f < \varepsilon\ .$$

This inequality is valid for every $\varepsilon > 0$, so according to the previous statement we have $\overline{\int_a^b} f = \underline{\int_a^b} f \in \mathbb{R}$. Then f has R. integral on $[a,b]$. ☐

We state another criterion of integrability without a proof.

Theorem 15 (Lebesgue characterisation of integrable functions)**.** *A function $f:\ [a,b] \to \mathbb{R}$ has Riemann integral, if and only if it is bounded and the set of its point of discontinuity on $[a,b]$ has* measure zero*.*

We define sets of measure zero as follows. A set $M \subset \mathbb{R}$ has *(Lebesgue) measure zero*, if for every $\varepsilon > 0$, there exists a sequence of intervals $I_1, I_2, \ldots$ such that

$$\sum_{i=1}^{\infty} |I_i| < \varepsilon \ \text{ and } \ M \subset \bigcup_{i=1}^{\infty} I_i\ .$$

In other words, M can be covered by intervals of arbitrarily small length. Simple properties of sets with measure zero:

- Every countable or finite set has measure zero.
- Every subset of a set of measure zero has measure zero.
- If each of countably many sets $A_1, A_2, \dots$ has measure zero, their union

$$\bigcup_{n=1}^{\infty} A_n$$

  has measure zero.
- Interval of positive length does not have measure zero.

For example, the set of rational numbers $\mathbb{Q}$ has measure zero. There exist sets of measure which are uncountable, classical example is *Cantor set.*

Theorem 16 (Monotonicity $\Rightarrow$ integrability)**.** *If $f : [a,b] \to \mathbb{R}$ is non-decreasing or non-increasing on $[a,b]$ then it is Riemann integrable.*

Proof. Assume that f is non-decreasing (for non-increasing f the argument is similar). For each subinterval $[\alpha, \beta] \subset [a,b]$ we have $\inf_{[\alpha,\beta]} f = f(\alpha)$ and $\sup_{[\alpha,\beta]} f = f(\beta)$. Given $\delta > 0$, we take any partition $D = (a_0, a_1, \dots, a_{k-1})$ interval $[a,b]$ with $\lambda(D) < \delta$ and

$$\begin{aligned} S(f,D) - s(f,D) &= \sum_{i=1}^{k} (a_i - a_{i-1})(\sup_{I_i} f - \inf_{I_i} f) \\ &= \sum_{i=1}^{k} (a_i - a_{i-1})(f(a_i) - f(a_{i-1})) \\ &\leq \delta \sum_{i=1}^{k} (f(a_i) - f(a_{i-1})) \\ &= \delta(f(a_k) - f(a_0)) = \delta(f(b) - f(a)) \, . \end{aligned}$$

This can be made arbitrarily small by reducing δ, in particular, given ε, choosing $\delta < \varepsilon/(f(b) - f(a))$ ensures that $S(f,D) - s(f,D) < \varepsilon$. Then, by the integrability criterion, $f \in \mathcal{R}(a,b)$. $\square$

Continuity is also sufficient for integrability. But we need to introduce its stronger form. Let us say that the function $f : I \to \mathbb{R}$, where I is the interval, is *uniformly continuous* (on I) if

$$\forall \varepsilon > 0 \; \exists \delta > 0 : \forall \; x, x' \in I, \; |x - x'| < \delta \Rightarrow |f(x) - f(x')| < \varepsilon \, .$$

That is, we require that single $\delta > 0$ works for all pairs of points x, x' in I. In the usual definition of continuity can δ depend on x. Uniform continuity implies continuity, but the reverse does not generally apply. For example, function

$$f(x) = 1/x : \; I = (0,1)$$

is continuous on i, but not uniformly continuous: $f(1/(n+1)) - f(1/n) = 1$, although $1/(n+1) - 1/n \to 0$ for $n \to \infty$. On a *compact interval* I, which is the interval of type $[a,b]$ where $-\infty < a \leq b < +\infty$, types of continuity coincide.

Theorem 17 (On compact: continuity $\Rightarrow$ uniform continuity)**.** *If the function $f: \; [a,b] \to \mathbb{R}$ on the interval $[a,b]$ is continuous, it is uniformly continuous.*

Proof. For contradiction, assume that $f: \; [a,b] \to \mathbb{R}$ is continuous at every point of the interval $[a,b]$ (i.e. one sided in the end points of a a b), but that it is not uniformly continuous on $[a,b]$. Negation of a uniform continuity means, that

$$\exists \varepsilon > 0 \; \forall \delta > 0 \; \exists x, x' \in I : |x - x'| < \delta \; \& \; |f(x) - f(x')| \geq \varepsilon \; .$$

Which means that there are points $x_n, x'_n \in [a,b]$ for $\delta = 1/n$ and $n = 1,2,\ldots$ that $|x_n - x'_n| < 1/n$, but $|f(x_n) - f(x'_n)| \geq \varepsilon$. Then, by Bolzano–Weierstrass theorem there exist subsequences of (x_n) and (x'_n) which both converge and (inevitably) have the same point α from $[a,b]$. This theorem asserts that there exists a sequence of indices $k_1 < k_2 < \ldots$ such that (x_{k_n}) converges. Again by the theorem there exists sequence of indices of $l_1 < l_2 < \ldots$ that $(x'_{k_{l_n}})$ converges. The sequence $(x_{k_{l_n}})$ remains convergent, because it is a subsequence of sequence (x_{k_n}). Because $|x_{k_{l_n}} - x'_{k_{l_n}}| < 1/k_{l_n} \leq 1/n \to 0$,

$$\lim_{n\to\infty} x_{k_{l_n}} = \lim_{n\to\infty} x'_{k_{l_n}} = \alpha \; .$$

To avoid multilevel indices, we rename $x_{k_{l_n}}$ to x_n and $x'_{k_{l_n}}$ to x'_n.) By Heine definition of limit, continuity of f in α and arithmetic of limits, we have

$$0 = f(\alpha) - f(\alpha) = \lim f(x_n) - \lim f(x'_n) = \lim (f(x_n) - f(x'_n)) \; .$$

This contradicts that $|f(x_n) - f(x'_n)| \geq \varepsilon$ for every n. □

Theorem 18 (Continuity $\Rightarrow$ integrability)**.** *If $f: \; [a,b] \to \mathbb{R}$ on the interval $[a,b]$ is continuous then it is Riemann integrable.*

Proof. Let f be continuous on $[a,b]$. Let $\varepsilon > 0$ be given. By the previous statement, we take $\delta > 0$ such that $|f(x) - f(x')| < \varepsilon$ when distance between $x, x' \in [a,b]$ is less than δ. Then

$$\sup_{[\alpha,\beta]} f - \inf_{[\alpha,\beta]} f \leq \varepsilon$$

for each subinterval $[\alpha, \beta] \subset [a, b]$ of length less than δ (why?). We take any partition $D = (a_0, a_1, \ldots, a_{k-1})$ of interval $[a, b]$ with $\lambda(D)$. We have that

$$\begin{aligned} S(f, D) - s(f, D) &= \sum_{i=1}^{k} (a_i - a_{i-1})(\sup_{I_i} f - \inf_{I_i} f) \\ &\le \sum_{i=1}^{k} (a_i - a_{i-1})\varepsilon \\ &= \varepsilon(a_k - a_0) = \varepsilon(b - a) \ . \end{aligned}$$

As in the previous theorem, the $\varepsilon(b - a)$ can be made arbitrarily small by reducing ε. Thus, according to the integrability criterion, $f \in \mathcal{R}(a, b)$. $\square$

Theorem 19 (Linearity of Riemann integral)**.**

(i) (linearity w.r.t. integrand) Let $f, g \in \mathcal{R}(a, b)$ be two functions having R. integrals and $\alpha, \beta \in \mathbb{R}$. Then

$$\alpha f + \beta g \in \mathcal{R}(a, b) \ \textit{and} \ \int_a^b (\alpha f + \beta g) = \alpha \int_a^b f + \beta \int_a^b g \ .$$

(ii) (linearity w.r.t.o boundaries) Let $f : [a, b] \to \mathbb{R}$ be a function and $c \in (a, b)$. Then

$$f \in \mathcal{R}(a, b) \iff f \in \mathcal{R}(a, c) \ \& \ f \in \mathcal{R}(c, b)$$

and, if these integrals are defined,

$$\int_a^b f = \int_a^c f + \int_c^b f \ .$$

If $a > b$, we define $\int_a^b f = -\int_b^a f$.

Corollary 20 ($\int$ over a cycle is 0)**.** *Let $a, b, c \in \mathbb{R}$, $d = \min(a, b, c)$, $e = \max(a, b, c)$ and $f \in \mathcal{R}(d, e)$. Then the following three integrals exist and satisfy*

$$\int_a^b f + \int_b^c f + \int_c^a f = 0 \ .$$

Lecture 5 (20.3.2019)

(translated and slightly adapted from lecture notes by Martin Klazar)

Theorem 21 (1st Fundamental Theorem of Calculus)**.** *Let $f \in \mathcal{R}(a,b)$ and function $F\colon\ [a,b] \to \mathbb{R}$ be defined as*

$$F(x) = \int_a^x f\ .$$

Then

(i) F is continuous on $[a,b]$ and

(ii) at every point of continuity $x_0 \in [a,b]$ of f there exists finite derivative $F'(x_0)$ and $F'(x_0) = f(x_0)$ (this applies one-sided if $x_0 = a$ or $x_0 = b$).

Proof. Let $c > 0$ be the upper bound for $|f(x)|$, $a \le x \le b$ (f is integrable and therefore bounded). For every two points $x, x_0 \in [a,b]$ we have

$$|F(x) - F(x_0)| = \Big| \int_a^x f - \int_a^{x_0} f \Big| = \Big| \int_{x_0}^x f \Big| \le |x - x_0| c\ ,$$

according to the definition of F, linearity $\int$ in integration limits and estimate $\int$ by upper sum for a trivial partition of the interval with end points x and x_0. Thus, for $x \to x_0$, we have $F(x) \to F(x_0)$. Therefore, F is continuous in x_0.

Let $x_0 \in [a,b]$ be a point of continuity of f. We have $\delta > 0$ that $f(x_0) - \varepsilon < f(x) < f(x_0) + \varepsilon$ once $|x - x_0| < \delta$. For $0 < x - x_0 < \delta$ then

$$f(x_0) - \varepsilon \le \frac{\int_{x_0}^x f}{x - x_0} = \frac{F(x) - F(x_0)}{x - x_0} \le f(x_0) + \varepsilon\ ,$$

according to the trivial estimate of $\int_{x_0}^x f$ by lower and upper sums for trivial partition (x_0, x). For $-\delta < x - x_0 < 0$ the same inequalities apply (both the numerator and the denominator of the fraction will change sign). For $x \to x_0$, $x \ne x_0$, we have $\frac{F(x) - F(x_0)}{x - x_0} \to f(x_0)$, or $F'(x_0) = f(x_0)$. $\square$

Corollary 22 (Continuous function has a primitive function)**.** *If $f\colon\ [a,b] \to \mathbb{R}$ is continuous on $[a,b]$, then f has a primitive function F on $[a,b]$.*

Proof. Just use the previous theorem and let $F(x) = \int_a^x f$. $\square$

Theorem 23 (2nd Fundamental Theorem of Calculus)**.** *If $f \in \mathcal{R}(a,b)$ and $F\colon\ [a,b] \to \mathbb{R}$ is primitive to f on $[a,b]$, then*

$$\int_a^b f = F(b) - F(a)\ .$$

Proof. Let $D = (a_0, a_1, \dots, a_k)$ be any partition of $[a, b]$. Using Lagrange's mean value theorem for each interval $I_i = [a_{i-1}, a_i]$ and the function F , we get

$$F(b) - F(a) = \sum_{i=1}^{k} (F(a_i) - F(a_{i-1})) = \sum_{i=1}^{k} f(c_i)(a_i - a_{i-1}) ,$$

for some points $a_i < c_i < a_{i+1}$ (since $F'(c_i) = f(c_i)$). Thus, (since $\inf_{I_i} f \leq f(c_i) \leq \sup_{I_i} f$)

$$s(f, D) \leq F(b) - F(a) \leq S(f, D) .$$

Then, from integrability of f, it follows that $F(b) - F(a) = \int_a^b f$. □

For a function $F : [a, b] \to \mathbb{R}$ we denote the difference of functional values in endpoints of the interval by

$$F|_a^b := F(b) - F(a) .$$

Previous results put together yield the following.

Corollary 24 ($\int$ and primitive function)**.** *If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$, then $f \in \mathcal{R}(a, b)$, f has a primitive function F on $[a, b]$ and*

$$\int_a^b f = F|_a^b = F(b) - F(a) .$$

Newton integral.

Let $f : (a, b) \to \mathbb{R}$ be such that a primitive function F of f on (a, b) has one sided limits $F(a^+) = \lim_{x \to a^+} F(x)$ a $F(b^-) = \lim_{x \to b^-} F(x)$. We define Newton integral of f on (a, b) as

$$(N) \int_a^b f = F(b^-) - F(a^+) .$$

Since different primitive functions of f differ by an additive constant, this difference does not depend on the choice of F and the definition is correct. The set of functions which are Newton integrable on (a, b) is denoted by $\mathcal{N}(a, b)$. We denote by $C(a, b)$ the set of functions continuous on $[a, b]$.

Theorem 25 (comparison of Newton and Riemann $\int$)**.**

(i) $C(a, b) \subset \mathcal{N}(a, b) \cap \mathcal{R}(a, b)$.

(ii) If $f \in \mathcal{N}(a, b) \cap \mathcal{R}(a, b)$, *then*

$$(N) \int_a^b f = (R) \int_a^b f .$$

(iii) The sets $\mathcal{N}(a, b) \backslash \mathcal{R}(a, b)$ *and* $\mathcal{R}(a, b) \backslash \mathcal{N}(a, b)$ *are nonempty.*

Proof. If f is continuous on $[a,b]$, by theorem from previous lecture, $f \in \mathcal{R}(a,b)$ and by First fundamental theorem of calculus, $F(x) = \int_a^x f$ is a primitive function to f on $[a,b]$. We have $F(a^+) = F(a) = 0$ a $F(b^-) = F(b) = \int_a^b f$, thus $f \in \mathcal{N}(a,b)$.

Let $f \in \mathcal{N}(a,b) \cap \mathcal{R}(a,b)$. Since $f \in \mathcal{N}(a,b)$, f has a primitive function F on (a,b) with one sided limits $F(a^+)$ and $F(b^-)$. Since $f \in \mathcal{R}(a,b)$, $f \in \mathcal{R}(a+\delta, b-\delta)$ for every $\delta > 0$ and by Second fundamental theorem of calculus we have

$$(R)\int_{a+\delta}^{b-\delta} f = F(b-\delta) - F(a+\delta) \, .$$

For $\delta \to 0^+$ the left hand side tends to $(R)\int_a^b f$ (f is bounded on $[a,b]$, thus integrals of f on $[a, a+\delta]$ and $[b-\delta, b]$ tend to 0) and left hand side tends to $F(b^-) - F(a^+) = (N)\int_a^b f$.

Function $f(x) = x^{-1/2} : \ (0,1] \to \mathbb{R}$, $f(0) = 42$, has Newton integral on $(0,1)$: $F(x) = 2x^{1/2}$ is primitive function of f on $(0,1)$, $F(0^+) = 0$ and $F(1^-) = 2$, thus $(N)\int_0^1 f = 2$. However, f is not bounded on $[0,1]$ and therefore $f \notin \mathcal{R}(0,1)$. Function $\operatorname{sgn}(x)$ is non-decreasing on $[-1,1]$ and thus Riemann integrable on $[-1,1]$. On the other hand, $\operatorname{sgn}(x)$ does not have Newton integral on $(-1,1)$ — as we showed on the first lecture, $\operatorname{sgn}(x)$ does not have a primitive function on $(-1,1)$. $\square$

Next we state variants of methods of computing primitive functions for definite integrals.

Theorem 26 (Integration by parts for definite integral)**.** *Let $f, g : [a,b] \to \mathbb{R}$ be functions with continuous derivatives f' and g' on $[a,b]$. Then,*

$$\int_a^b fg' = fg|_a^b - \int_a^b f'g \, .$$

Theorem 27 (Substitution for definite integral)**.** *Let $\varphi : \ [\alpha,\beta] \to [a,b]$ and $f : \ [a,b] \to \mathbb{R}$ are two functions such that φ has continuous derivative on $[\alpha,\beta]$ and $\varphi(\alpha) = a$, $\varphi(\beta) = b$ or $\varphi(\alpha) = b$, $\varphi(\beta) = a$. If*

(i) f is continous on $[a,b]$, or

(ii) if φ is strictly monotonous on $[\alpha,\beta]$ and $f \in \mathcal{R}(a,b)$

then

$$\int_\alpha^\beta f(\varphi)\varphi' = \int_{\varphi(\alpha)}^{\varphi(\beta)} f = \begin{cases} \int_a^b f \quad \text{or} \\ \int_b^a f = -\int_a^b f \, . \end{cases}$$

Proof of (i). The function f is continuous, so it has a primitive function F. Derivative of a composed function $F(\varphi)$ on $[\alpha,\beta]$ is $F(\varphi)' = f(\varphi)\varphi'$. So, $F(\varphi)$

is on $[\alpha, \beta]$ a primitive function of $f(\varphi)\varphi'$. The function $f(\varphi)\varphi'$ is continuous (since product of two continuous functions is continuous) on $[\alpha, \beta]$, thus, $f(\varphi)\varphi' \in \mathcal{R}(\alpha, \beta)$. Thus, applying 2nd fundamental theorem of calculus twice (the first and the third equality), we have

$$\int_{\alpha}^{\beta} f(\varphi)\varphi' = F(\varphi)|_{\alpha}^{\beta} = F|_{\varphi(\alpha)}^{\varphi(\beta)} = \int_{\varphi(\alpha)}^{\varphi(\beta)} f \ .$$

$\square$

Lecture 6 (27.3.2019)

(translated and slightly adapted from lecture notes by Martin Klazar)

Applications of integrals

We estimate *factorial* $n! = 1 \cdot 2 \cdot \ldots \cdot n$ as follows: for $f(x) = \log x : [1, +\infty) \to [0, +\infty)$ and a partition $D = (1, 2, \ldots, n+1)$ of interval $[1, n+1]$ we have

$$s(f, D) = \sum_{i=1}^{n} 1 \cdot \log i = \log(n!) \quad \text{a} \quad S(f, D) = \sum_{i=1}^{n} 1 \cdot \log(i+1) = \log((n+1)!) \ .$$

Since $s(f, D) < \int_1^{n+1} \log x = (n+1)\log(n+1) - (n+1) + 1 < S(f, D)$, for $n \geq 2$ we get estimate

$$n \log n - n + 1 < \log(n!) < (n+1)\log(n+1) - n$$

and so

$$e\left(\frac{n}{e}\right)^n < n! < e\left(\frac{n+1}{e}\right)^{n+1} \ .$$

Similarly we estimate *harmonic numbers* H_n,

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} \ .$$

For a function $f(x) = 1/x : (0, +\infty) \to (0, +\infty)$ and a partition $D = (1, 2, \ldots, n+1)$ of interval $[1, n+1]$ we have that

$$s(f, D) = \sum_{i=1}^{n} 1 \cdot \frac{1}{i+1} = H_{n+1} - 1 \quad \text{a} \quad S(f, D) = \sum_{i=1}^{n} 1 \cdot \frac{1}{i} = H_n \ .$$

Since $s(f, D) < \int_1^{n+1} 1/x = \log(n+1) < S(f, D)$, for $n \geq 2$ we get

$$\log(n+1) < H_n < 1 + \log n \ .$$

Similarly one can estimate also sums of infinite series, but we need integral over infinite domain to do that.

For $a \in \mathbb{R}$ and $f : [a, +\infty) \to \mathbb{R}$ such that $f \in \mathcal{R}(a, b)$ for every $b > a$, we define

$$\int_a^{+\infty} f := \lim_{b \to +\infty} \int_a^b f \ ,$$

if the limit exists (we allow $\pm\infty$). We say that the integral converges if and only if the limit is a real number and we say that the integral diverges otherwise.

Theorem 28 (Integral criterion of convergence)**.** *Let a be and integer and $f : [a, +\infty) \to \mathbb{R}$ be a function which is non-negative and non-increasing on $[a, +\infty)$. Then,*

$$\sum_{n=a}^{\infty} f(n) = f(a) + f(a+1) + f(a+2) + \ldots < +\infty \iff \int_a^{+\infty} f < +\infty \, .$$

So, the series converges if and only if the corresponding integral converges.

Proof. The sequence of partial sums of the series is non-decreasing and therefore it has a limit which is either real or $+\infty$. Since f is monotone, $f \in \mathcal{R}(a, b)$ for every real $b > a$. Moreover, since f is non-negative, $\int_a^{b'} f \geq \int_a^b f$, if $b' \geq b$. Then $\lim_{b\to+\infty} \int_a^b f$ exists and is either real or $+\infty$. For some integer $b > a$, consider the partition $D = (a, a+1, a+2, \ldots, b)$ of $[a, b]$. We have the following inequalities:

$$\sum_{i=a+1}^{b} f(i) = s(f, D) \leq \int_a^b f \leq S(f, D) = \sum_{i=a}^{b-1} f(i) \, .$$

It follows that bounded partial sums imply bounded integrals $\int_a^b f$ for any integer $b > a$ and the other way round. Thus, both limits are either real or $+\infty$. ∎

Now we can easily decide convergence of

$$\sum_{n=1}^{\infty} \frac{1}{n^s}, \; s > 0 \, .$$

For $s \neq 1$, we have

$$\int_1^{+\infty} \frac{dx}{x^s} = \left. \frac{x^{1-s}}{1-s} \right|_1^{+\infty} = (1-s)^{-1}(\lim_{x\to+\infty} x^{1-s} - 1) \, ,$$

this equals $+\infty$ for $0 < s < 1$ and $(s-1)^{-1}$ for $s > 1$. For $s = 1$ we have

$$\int_1^{+\infty} \frac{dx}{x} = \log x|_1^{+\infty} = \lim_{x\to+\infty} \log x = +\infty \, .$$

Thus, by integral criterion the series converges if and only if $s > 1$.

Next, consider the series

$$\sum_{n=2}^{\infty} \frac{1}{n \log n} \, .$$

Here,

$$\int_2^{+\infty} \frac{dx}{x \log x} = \log\log x|_2^{+\infty} = \lim_{x\to+\infty} \log\log x - \log\log 2 = +\infty \, .$$

By integral criterion the series diverges. Exercise: analyze convergence of $\sum_{n\geq 2} 1/n(\log n)^s$, $s > 1$.

We have already shown estimates of factorial using integrals. Now we show how to extend factorial to a smooth function on $[1, +\infty)$.

Theorem 29 (Gamma function)**.** *Function* Γ *defined as*

$$\Gamma(x) := \int_0^{+\infty} t^{x-1}e^{-t}\, dt : \ [1, +\infty) \to (0, +\infty)$$

satisfies the following functional equation

$$\Gamma(x+1) = x\Gamma(x) \ .$$

on interval $[1, +\infty)$*. Moreover,* $\Gamma(1) = 1$ *and* $\Gamma(n) = (n-1)!$ *for integers* $n \geq 2$*.*

Proof. First, we show that $\Gamma(x)$ is correctly defined. For every fixed $x \geq 1$, the integrand is a non-negative continuous function (for $x = 1$ and $t = 0$ we let $0^0 = 1$). Since $\lim_{t\to+\infty} t^{x-1}e^{-t/2} = 0$ (exponential grows faster than a polynomial), for every $t \in [0, +\infty)$ we have the following inequality:

$$t^{x-1}e^{-t} = t^{x-1}e^{-t/2} \cdot e^{-t/2} \leq ce^{-t/2} \ ,$$

where $c > 0$ is a constant depending only on x. Thus, integrals over finite intervals $[0, b]$ are defined, for $b \to +\infty$ don't decrease and have a finite limit:

$$\int_0^b t^{x-1}e^{-t}\, dt \leq \int_0^b ce^{-t/2}\, dt = c(1 - e^{-b/2}/2) < c \ .$$

The value $\Gamma(x)$ is therefore defined for every $x \geq 1$. For $x = 1$, we have

$$\Gamma(1) = \int_0^{+\infty} e^{-t}\, dt = (-e^{-t})|_0^{+\infty} = 0 - (-1) = 1 \ .$$

Functional equation can be derived by integration per partes:

$$\begin{aligned}
\Gamma(x+1) &= \int_0^{+\infty} t^x e^{-t}\, dt = t^x(-e^{-t})|_0^{+\infty} - \int_0^{+\infty} xt^{x-1}(-e^{-t})\, dt \\
&= 0 - 0 + x\int_0^{+\infty} t^{x-1}e^{-t}\, dt \\
&= x\Gamma(x) \ .
\end{aligned}$$

Values $\Gamma(n)$ follow by induction. $\square$

Note that extending factorial to a function f on $[1, +\infty)$ satisfying $f(x + 1) = xf(x)$ can be done in many ways, starting from any function defined on

$[1,2)$ with $f(1)=1$ and extending it. The advantage of $\Gamma(x)$ is that it has derivatives of all orders.

Finally, we give formulas for area, length of a curve and volume of solids of revolution. We have essentially defined area $U(a,b,f)$ (that is, points (x,y) in a plane satisfying $a\le x\le b$ a $0\le y\le f(x)$) under the graph of function f as $\int_a^b f$.

For a function $f:\ [a,b]\to\mathbb{R}$ we define length of its graph $G=\{(x,f(x))\in\mathbb{R}^2\mid a\le x\le b\}$ as a limit of length of a sequence of broken lines L with endpoints of segments on G which "approximate G", where the length of a longest segment of L tends to 0. For "nice" functions f (for instance those with continuous derivative), this limit exists and we can calculate it using Riemann integral. A segment of L connecting points $(x,f(x))$ and $(x+\Delta,f(x+\Delta))$ has by Pythagoras theorem length

$$\sqrt{\Delta^2+(f(x+\Delta)-f(x))^2}=\Delta\sqrt{1+\left(\frac{f(x+\Delta)-f(x)}{\Delta}\right)^2}\,.$$

From this, one can derive the following formula:

Theorem (length of a curve). *Let $f:\ [a,b]\to\mathbb{R}$ be a function with continuous derivative on $[a,b]$ (so $\sqrt{1+(f')^2}\in\mathcal{R}(a,b)$). Then*

$$\mathrm{length}(\{(x,f(x))\in\mathbb{R}^2\mid a\le x\le b\})=\int_a^b\sqrt{1+(f'(t))^2}\,dt\ .$$

For a subset $M\subset\mathbb{R}^3$ we can define its volume as a limit, for $n\to\infty$, of the sume of volumes of $1/n^3$ cubes K in the set

$$\{K=[\tfrac{a}{n},\tfrac{a+1}{n}]\times[\tfrac{b}{n},\tfrac{b+1}{n}]\times[\tfrac{c}{n},\tfrac{c+1}{n}]\mid a,b,c\in\mathbb{Z}\ \&\ K\subset M\}\ .$$

If M is "nice", this limit exists and can be computed using integral. In particular, if M is obtained by rotating some planar figure around the horizontal axis, we get the following.

Theorem (volume of solid of revolution). *Let $f\in\mathcal{R}(a,b)$ and $f\ge 0$ on $[a,b]$. For a volume of a body defined as*

$$V=\{(x,y,z)\in\mathbb{R}^3\mid a\le x\le b\ \&\ \sqrt{y^2+z^2}\le f(x)\}$$

obtained by rotating a planar figure $U(a,b,f)$ under the graph of a function f around x-axis we have

$$\mathrm{volume}(V)=\pi\int_a^b f(t)^2\,dt\ .$$

The formula can be obtained by cutting V by planes perpendicular to x-axis into slices of length $\Delta>0$ and summing their volumes. Each slice is roughly a cylinder with radius $|f(x)|$ and height Δ.

Lecture 7 (3.4.2019)

(partially translated and adapted from lecture notes by Martin Klazar)

Multivariable calculus

We will work in m-dimensional *Euclidean space* $\mathbb{R}^m$, $m \in \mathbb{N}$, which is a set of all ordered m-tuples of reals $\mathbf{x} = (x_1, x_2, \ldots, x_m)$ with $x_i \in \mathbb{R}$. It is an m-dimensional vector space over $\mathbb{R}$ — we can sum and subtract its elements and we can multiply them by real constants. We introduce a notion of distance in $\mathbb{R}^m$, using *(Euclidean) norm* wich is a mapping $\|\cdot\|: \mathbb{R}^m \to [0, +\infty)$ defined as

$$\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2 + \ldots + x_m^2}\ .$$

Euclidean norm has the following properties ($a \in \mathbb{R}$, $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$):

(i) (positivity) $\|\mathbf{x}\| \geq 0$ a $\|\mathbf{x}\| = 0 \iff \mathbf{x} = \mathbf{o} = (0, 0, \ldots, 0)$,

(ii) (homogenity) $\|a\mathbf{x}\| = |a| \cdot \|\mathbf{x}\|$ and

(iii) (triangle inequality) $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$.

Using the norm, we define *(Euclidean) distance* $d(\mathbf{x}, \mathbf{y}): \mathbb{R}^m \times \mathbb{R}^m \to [0, +\infty)$ between two points x and y in $\mathbb{R}^m$ as

$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \ldots + (x_m - y_m)^2}\ .$$

Properties of Euclidean distance ($\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^m$):

(i) (positivity) $d(\mathbf{x}, \mathbf{y}) \geq 0$ and $d(\mathbf{x}, \mathbf{y}) = 0 \iff \mathbf{x} = \mathbf{y}$,

(ii) (symmetry) $d(\mathbf{x}, \mathbf{y}) = d(\mathbf{y}, \mathbf{x})$ and

(iii) (triangle inequality) $d(\mathbf{x}, \mathbf{y}) \leq d(\mathbf{x}, \mathbf{z}) + d(\mathbf{z}, \mathbf{y})$.

With exception of triangle inequality (deriving of which requires more effort), these properties of norm and distance follow easily form the definition.

(Open) ball $B(\mathbf{a}, r)$ with radius $r > 0$ and center $\mathbf{a} \in \mathbb{R}^m$ is the set of points in $\mathbb{R}^m$ with distance from $\mathbf{a}$ less than r:

$$B(\mathbf{a}, r) = \{\mathbf{x} \in \mathbb{R}^m \mid \|\mathbf{x} - \mathbf{a}\| < r\}\ .$$

Open set in $\mathbb{R}^m$ is a subset $M \subset \mathbb{R}^m$ such that for every point $\mathbf{x} \in M$ there is a ball with center $\mathbf{x}$ contained in M:

$$M \text{ is open} \iff \forall \mathbf{x} \in M\ \exists r > 0:\ B(\mathbf{x}, r) \subset M\ .$$

Following properties of open sets in $\mathbb{R}^m$ can be derived as a simple exercise:

(i) sets $\emptyset$ a $\mathbb{R}^m$ are open,

(ii) union $\bigcup_{i\in I} A_i$ of any system $\{A_i \mid i \in I\}$ of open sets A_i is an open set

(iii) intersection of two (finitely many) open sets is an open set.

Intersection of infinitely many open sets might not be open. *Neighborhood of a point* $\mathbf{a} \in \mathbb{R}^m$ is any open set in $\mathbb{R}^m$ containing $\mathbf{a}$.

We will consider functions $f:\ M \to \mathbb{R}$, $f = f(x_1, x_2, \ldots, x_m)$, defined on $M \subset \mathbb{R}^m$ and mappings

$$f:\ M \to \mathbb{R}^n,\ M \subset \mathbb{R}^m,\ f = (f_1, f_2, \ldots, f_n)\ ,$$

where $f_i = f_i(x_1, x_2, \ldots, x_m)$ are *coordinate functions*. Our goal will be to generalize derivative as a linear approximation and a notion of integral to functions of several variables.

First, we generalize concept of continuity. Let $U \subset \mathbb{R}^m$ be a neighborhood of a point $\mathbf{a} \in \mathbb{R}^m$. We say that a function $f:\ U \to \mathbb{R}$ is *continuous at* $\mathbf{a}$, if

$$\forall\varepsilon > 0\ \exists\delta > 0:\ \ \|\mathbf{x} - \mathbf{a}\| < \delta \Rightarrow |f(\mathbf{x}) - f(\mathbf{a})| < \varepsilon\ .$$

More generally, a mapping $f:\ U \to \mathbb{R}^n$, is *continuous at* $\mathbf{a}$ if

$$\forall\varepsilon > 0\ \exists\delta > 0:\ \ \|\mathbf{x} - \mathbf{a}\| < \delta \Rightarrow \|f(\mathbf{x}) - f(\mathbf{a})\| < \varepsilon\ ,$$

i.e., we replace absolute value (which is the norm in $\mathbb{R}^1$) by norm in $\mathbb{R}^n$. Similarly, we can generalize the notion of limit of a function:

$$\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = c \Leftrightarrow \forall\varepsilon > 0\ \exists\delta > 0 : \mathbf{x} \in B(\mathbf{a}, \delta) \setminus \{\mathbf{a}\} \Rightarrow |f(\mathbf{x}) - c| < \varepsilon.$$

Multivarible Riemann Integral

First, we generalize a notion of Riemann integral to multivariable functions, defining multivariable analogues of partition of an interval and upper and lower Riemann sum.

An *n-dimensional box* is a Cartesian product of closed intervals

$$I = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$$

where $-\infty < a_i < b_i < \infty, i = 1, \ldots, n$. For instance, for 2-dimensional box is a closed rectangle with sides parallel to the axes.

Volume of a box is defined as $|I| = \prod_{i=1}^n (b_i - a_i)$. A *partition* of a box is a set of boxes

$$D = \{[c_1^{j_1}, c_1^{j_1+1}] \times \cdots \times [c_n^{j_n}, c_n^{j_n+1}] | 0 \le j_i < k_i, 1 \le i \le n\},$$

where $a_i = c_i^0 < c_i^1 < \cdots < c_i^{k_i} = b_i$ are some partitions of the intervals $[a_i, b_i]$, $i = 1, \ldots, n$. *Norm* of a partition is defined as

$$\lambda(D) = \max_{0\le j_i<k_i, 1\le i\le n} (c_i^{j+1} - c_i^j),$$

i.e., as a maximal "length of an edge of a sub-box".

One can now define a partition with points and generalize a Riemann definition of integral. However, we will proceed by generalizing Darboux definition of the integral.

Let I be a box with a partition D and let $f : I \to \mathbb{R}$ be a function. For every box $J \in D$ we define $m(J) = \inf_{\mathbf{x}\in J} f(\mathbf{x})$ and $M(J) = \sup_{\mathbf{x}\in J} f(\mathbf{x})$. We define lower and upper Riemann sum as

$$s(f,D) = \sum_{J\in D} |J| \cdot m(J), \quad S(f,D) = \sum_{J\in D} |J| \cdot M(J)$$

and lower and upper integral as

$$\underline{\int_I} f = \sup(\{s(f,D) | D \text{ is a partition of } I\}),$$

$$\overline{\int_I} f = \inf(\{S(f,D) | D \text{ is a partition of } I\}).$$

Similarly as in one dimension, the following inequalities hold

$$s(f,D) \le \underline{\int_I} f \le \overline{\int_I} f \le S(f,D).$$

Integral of f on I is then defined as a real number

$$\int_I f = \underline{\int_I} f = \overline{\int_I} f$$

if upper integral equals lower integral.

We denote the set of functions which have integral on I by $\mathcal{R}(I)$.

We say that a set $E \subseteq \mathbb{R}^m$ has *measure zero* if for every $\varepsilon > 0$ exists a sequence of boxes $I_1, I_2, \dots$ in $\mathbb{R}^m$, such that $\sum_{n=1}^{\infty} |I_n| < \varepsilon$ and $E \subset \cup_{n=1}^{\infty} I_n$.

Theorem 30. *Let $I \subseteq \mathbb{R}^m$ be a box and $f : I \to \mathbb{R}$ is a well defined function. Then $f \in \mathcal{R}(I)$ if and only if f is bounded and a set its points of discontinuity has measure zero.*

Integral over a bounded set $E \subset \mathbb{R}^m$ which is not a box: A *characteristic function* of a set E is a function $\chi_E : \mathbb{R}^m \to \{0,1\}$ defined as $\chi_e(\mathbf{x}) = 1$ if $\mathbf{x} \in E$ and $\chi_e(\mathbf{x}) = 0$ otherwise. Let I be a box containing E. *Volume* of E is defined as $vol(E) = \int_I \chi_E$, if the integral exists. Finally, we define $\int_E f = \int_I f(\mathbf{x}) \cdot \chi_E$.

Lecture 8 (10.4.2019)

(partially translated and adapted from lecture notes by Martin Klazar)

Multivariable calculus

The following theorem gives a method how to compute multivariable Riemann integral by computing "ordinary" integrals.

Theorem 31 (Fubini)**.** *Let $X \subset \mathbb{R}^m$, $Y \subset \mathbb{R}^n$ and $Z = X \times Y \subset \mathbb{R}^{m+n}$ be $m-$, $n-$, and $m+n$-dimensional boxes, respectively. Let $f : Z \to \mathbb{R}$, $f \in \mathcal{R}(Z)$. Then integrals $\int_Z f$, $\int_X(\int_Y f)$ and $\int_Y(\int_X f)$ exist and are all equal.*

Integrals $\int_X(\int_Y f)$ and $\int_Y(\int_X f)$ have the following meaning. Define a function $F : X \to \mathbb{R}$ as $F(\mathbf{x}) = \int_Y f(\mathbf{x}, \mathbf{y})\ d\mathbf{y}$, whenever $\int_Y f(\mathbf{x}, \mathbf{y})\ d\mathbf{y}$ exists and by arbitrary value from the interval $[\underline{\int_Y} f(\mathbf{x}, \mathbf{y})\ d\mathbf{y}, \overline{\int_Y} f(\mathbf{x}, \mathbf{y})\ d\mathbf{y}]$ otherwise. We then interpret $\int_X(\int_Y f)$ as $\int_X F$. We define a function $G : Y \to \mathbb{R}$ and interpret $\int_Y(\int_X f)$ analogously as $\int_Y G$.

By repeated application of Fubini Theorem, one can derive the following.

Corollary 32. *Let $I = [a_1, b_1] \times \cdots \times [a_n, b_n]$ be a box and let $f : I \to \mathbb{R}$, $f \in \mathcal{R}(I)$. Then*

$$\int_I f = \int_{a_n}^{b_n} \left(\int_{a_{n-1}}^{b_{n-1}} \cdots \left(\cdots \int_{a_1}^{b_1} f(x_1, \dots x_n)\ dx_1 \right) \cdots\ dx_{n-1} \right)\ dx_n.$$

Note that the order of variables can be chosen arbitrarily.

Directional derivative, partial derivative, total differential

Let $U \subset \mathbb{R}^m$ be a neighborhood of a point $\mathbf{a}$ and $f : U \to \mathbb{R}$ be a function. *Directional derivative* of f at a point $\mathbf{a}$ in direction $\mathbf{v} \in \mathbb{R}^m \backslash \{\mathbf{o}\}$ is defined as a limit

$$\mathrm{D}_{\mathbf{v}} f(\mathbf{a}) := \lim_{t \to 0} \frac{f(\mathbf{a} + t\mathbf{v}) - f(\mathbf{a})}{t}\ ,$$

if it exists. Imagine that U is an area in three dimensional Euclidean space, where f is a function of temperature in a given point and a particle moving through the area. Directional derivatives $\mathrm{D}_{\mathbf{v}} f(\mathbf{a})$ corresponds to immediate change of temperature of surroundings of a particle in a moment when it is at a point $\mathbf{a}$ and has velocity $\mathbf{v}$.

Partial derivative of a function f at a point $\mathbf{a}$ with respect to the i-th variable x_i is a directional derivative $\mathrm{D}_{\mathbf{e}_i} f(\mathbf{a})$, where $\mathbf{e}_i$ the i-th vector of canonical basis, i.e., $\mathbf{e}_i = (0, 0, \dots, 0, 1, 0, 0, \dots, 0)$ has i-th coordinate 1 and all other coordinates 0. We denote partial derivative by $\frac{\partial f}{\partial x_i}(\mathbf{a})$ (or, as a shortcut $\partial_i f(\mathbf{a})$. Thus, partial derivative equals to the following limit.

$$\frac{\partial f}{\partial x_i}(\mathbf{a}) = \lim_{h \to 0} \frac{f(a_1, \dots, a_{i-1}, a_i + h, a_{i+1}, \dots, a_m) - f(a_1, a_2, \dots, a_m)}{h} .$$

The vector of values of all partial derivatives of a function f at a point $\mathbf{a}$ is called the *gradient* of f at $\mathbf{a}$ and is denoted $\nabla f(\mathbf{a})$.

$$\nabla f(\mathbf{a}) := (\tfrac{\partial f}{\partial x_1}(\mathbf{a}), \tfrac{\partial f}{\partial x_2}(\mathbf{a}), \dots, \tfrac{\partial f}{\partial x_m}(\mathbf{a})) .$$

A function $f : U \to \mathbb{R}$, $U \subseteq \mathbb{R}^m$, is *differentable* at $\mathbf{a} \in U$ if there exists a linear mapping $L : \mathbb{R}^m \to \mathbb{R}$, such that

$$\lim_{\mathbf{h} \to \mathbf{o}} \frac{f(\mathbf{a} + \mathbf{h}) - f(\mathbf{a}) - L(\mathbf{h})}{\|\mathbf{h}\|} = 0 .$$

This mapping L is called *(total) differential (or total derivative)* of f at $\mathbf{a}$ and is denoted by $\mathrm{D}f(\mathbf{a})$.

More generally, a mapping $f : U \to \mathbb{R}^n$is *differentiable* at $\mathbf{a} \in U$, if there exists a linear mapping $L : \mathbb{R}^m \to \mathbb{R}^n$ satisfying

$$\lim_{\mathbf{h} \to \mathbf{o}} \frac{\|f(\mathbf{a} + \mathbf{h}) - f(\mathbf{a}) - L(\mathbf{h})\|}{\|\mathbf{h}\|} = 0$$

(note that norm in the norm in the denominator is in $\mathbb{R}^m$ and the norm in the numerator in $\mathbb{R}^n$). Again, we call L *differential* and denote it by $\mathrm{D}f(\mathbf{a})$. An important difference between directional and partial derivatives, which are simply real numbers, and the differential is, that the differential is a more complex object — a linear mapping.

Directional derivatives, partial derivatives and the total differential give the following linear approximations of f close to $\mathbf{a}$:

$$\begin{aligned} f(\mathbf{a} + t\mathbf{v}) &= f(\mathbf{a}) + \mathrm{D}_{\mathbf{v}} f(\mathbf{a}) \cdot t + o(t),\ t \to 0 , \\ f(\mathbf{a} + t\mathbf{e}_i) &= f(\mathbf{a}) + \frac{\partial f}{\partial x_i}(\mathbf{a}) \cdot t + o(t),\ t \to 0 , \\ f(\mathbf{a} + \mathbf{h}) &= f(\mathbf{a}) + \mathrm{D}f(\mathbf{a})(\mathbf{h}) + o(\|\mathbf{h}\|),\ \|\mathbf{h}\| \to 0 . \end{aligned}$$

In the first two expressions t is a real number and the approximation is relevant only for arguments on the line in the direction $\mathbf{v}$, in the third expression, $\mathbf{h} \in \mathbb{R}^m$ and approximation works for any argument close to $\mathbf{a}$.

Differentiability is a stronger property than existence of directional and partial derivatives. (Moreover, existence of all partial/directional derivatives at a point does not even imply continuity!)

One can calculate partial derivative with respect to x_i using the same methods as computing derivatives of functions of single variable — by treating all the variables except x_i as constants.

Theorem 33 (Properties of differential)**.** *Let $f = (f_1, f_2, \ldots, f_n) : U \to \mathbb{R}^n$ be a mapping and $U \subset \mathbb{R}^m$ a neighborhood of $\mathbf{a}$.*

1. *Differential of f at $\mathbf{a}$ is unique (if it exists).*
2. *A mapping f is differentiable at $\mathbf{a}$, if and only if each coordinate function f_i is differentiable at $\mathbf{a}$.*
3. *If f is differentiable at $\mathbf{a}$, then f is continuous at $\mathbf{a}$.*

Theorem 34 (Differential $\Rightarrow \partial$)**.** *Let $U \subset \mathbb{R}^m$ be a neighborhood of a point $\mathbf{a}$ and $f : U \to \mathbb{R}$ a function differentiable at $\mathbf{a}$. Then f has all partial derivatives at $\mathbf{a}$ and their values determine the differential:*

$$\begin{aligned} \mathrm{D}f(\mathbf{a})(\mathbf{h}) &= \frac{\partial f}{\partial x_1}(\mathbf{a}) \cdot h_1 + \frac{\partial f}{\partial x_2}(\mathbf{a}) \cdot h_2 + \cdots + \frac{\partial f}{\partial x_m}(\mathbf{a}) \cdot h_m \\ &= \langle \nabla f(\mathbf{a}), \mathbf{h} \rangle \end{aligned}$$

(i.e., value of the differential at $\mathbf{h}$ is a scalar product of $\mathbf{h}$ and a gradient of f at $\mathbf{a}$). Moreover, f then also has all directional derivatives at $\mathbf{a}$ and $\mathrm{D}_{\mathbf{v}} f(\mathbf{a}) = \mathrm{D}f(\mathbf{a})(\mathbf{v})$.

Proof. Since the differential is defined as a linear mapping $L = \mathrm{D}f(\mathbf{a})$, we have

$$L(\mathbf{h}) = L(h_1\mathbf{e}_1 + h_2\mathbf{e}_2 + \cdots + h_m\mathbf{e}_m) = L(e_1)h_1 + \cdots + L(e_m)h_m \ ,$$

where e_i is the i-th vector of the canonical base. Thus, by definition of total differential at $\mathbf{a}$, $f(\mathbf{a} + t\mathbf{e}_i) = f(\mathbf{a}) + L(t\mathbf{e}_i) + o(\|t\mathbf{e}_i\|)$ as $t \to 0$.

$$\begin{aligned} \frac{\partial f}{\partial x_i}(\mathbf{a}) &= \lim_{t\to 0} \frac{f(\mathbf{a} + t\mathbf{e}_i) - f(\mathbf{a})}{t} = \lim_{t\to 0} \frac{L(t\mathbf{e}_i) + o(\|t\mathbf{e}_i\|)}{t} \\ &= \lim_{t\to 0} \frac{tL(\mathbf{e}_i) + o(|t|)}{t} = L(\mathbf{e}_i) + \lim_{t\to 0} \frac{o(|t|)}{t} \\ &= L(\mathbf{e}_i) \ . \end{aligned}$$

Thus, $L(e_i) = \frac{\partial f}{\partial x_i}(a)$.

Let $\mathbf{v} \in \mathbb{R}^m$ be a nonzero vector. Since $\mathbf{v} = v_1\mathbf{e}_1 + \ldots + v_m\mathbf{e}_m$ and f has all partial derivatives at $\mathbf{a}$, by similar reasoning we get that $\mathrm{D}_{\mathbf{v}} f(\mathbf{a})$ equals to $\mathrm{D}f(\mathbf{a})(\mathbf{v})$. $\square$

The differential of a mapping $f : U \to \mathbb{R}^n$, a mapping $L = \mathrm{D}f(\mathbf{a}) : \mathbb{R}^m \to \mathbb{R}^n$, can be described by an $n \times m$ matrix, where $L(\mathbf{h})$ is the result of multiplication of $\mathbf{h}$ by the matrix:

$$L(\mathbf{h}) = \begin{pmatrix} L(\mathbf{h})_1 \\ L(\mathbf{h})_2 \\ \vdots \\ L(\mathbf{h})_n \end{pmatrix} = \begin{pmatrix} l_{1,1} & l_{1,2} & \ldots & l_{1,m} \\ l_{2,1} & l_{2,2} & \ldots & l_{2,m} \\ \vdots & \vdots & \ldots & \vdots \\ l_{n,1} & l_{n,2} & \ldots & l_{n,m} \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_m \end{pmatrix} .$$

where i-th row of this matrix is a gradient of the coordinate function f_i at a point $\mathbf{a}$:

$$l_{i,j} = \frac{\partial f_i}{\partial x_j}(\mathbf{a}) \, .$$

Corollary 35 (Jacobi matrix)**.** *Differential of a mapping $f : U \to \mathbb{R}^n$ at a point $\mathbf{a}$, where $U \subset \mathbb{R}^m$ is a neighborhood of $\mathbf{a}$ and f has coordinate functions $f = (f_1, f_2, \ldots, f_n)$, is determined by* Jacobi matrix *if the mapping f at a point $\mathbf{a}$:*

$$\left(\frac{\partial f_i}{\partial x_j}(\mathbf{a})\right)_{i,j=1}^{n,m} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{a}) & \frac{\partial f_1}{\partial x_2}(\mathbf{a}) & \ldots & \frac{\partial f_1}{\partial x_m}(\mathbf{a}) \\ \frac{\partial f_2}{\partial x_1}(\mathbf{a}) & \frac{\partial f_2}{\partial x_2}(\mathbf{a}) & \ldots & \frac{\partial f_2}{\partial x_m}(\mathbf{a}) \\ \vdots & \vdots & \ldots & \vdots \\ \frac{\partial f_n}{\partial x_1}(\mathbf{a}) & \frac{\partial f_n}{\partial x_2}(\mathbf{a}) & \ldots & \frac{\partial f_n}{\partial x_m}(\mathbf{a}) \end{pmatrix} .$$

If the Jacobi matrix is a square matrix, its determinant is called *jacobian.*

Theorem 36 ($\partial \Rightarrow$ differential)**.** *Let $U \subset \mathbb{R}^m$ is a neighborhood of a point $\mathbf{a} \in \mathbb{R}^m$. If a function $f : U \to \mathbb{R}$ has all partial derivatives on U and they are continuous at $\mathbf{a}$, then f is differentiable at $\mathbf{a}$.*

Geometry of partial derivatives and differentials.

We now generalize the notion of tangent line to a graph of a function of one variable to a tangent (hyper-)plane to a graph of a function of several variables. For simplicity, we consider only tangent planes for functions of two variables, general tangent hyperplanes are defined in an analogous way (but are hard to imagine).

Let $(x_0, y_0) \in U \subset \mathbb{R}^2$, where U is an open set in a plane, and $f : U \to \mathbb{R}$ is a function. Its graph

$$G_f = \{(x, y, z) \in \mathbb{R}^3 \mid (x, y) \in U, z = f(x, y)\}$$

is a surface in three dimensional Euclidean space. On G_f, there exists a point (x_0, y_0, z_0), such that $z_0 = f(x_0, y_0)$. Assume that f is differentiable at (x_0, y_0). Then, there exists a unique linear functions of two variables $L(x, y)$ (i.e. $L(x, y) = \alpha + \beta x + \gamma y$), such that a graph of $L(x, y)$ contains (x_0, y_0, z_0), and it satisfies

$$\lim_{(x,y)\to(x_0,y_0)} \frac{f(x, y) - L(x, y)}{d((x, y), (x_0, y_0))} = 0.$$

Specifically, it is a function

$$T(x, y) = z_0 + \frac{\partial f}{\partial x}(x_0, y_0) \cdot (x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0) \cdot (y - y_0) \, .$$

It follows from the uniqueness of a diffenential, because $T(x, y) = z_0 + \mathrm{D}f(x_0, y_0)(x - x_0, y - y_0)$. Graph of $T(x, y)$

$$G_T = \{(x, y, z) \in \mathbb{R}^3 \mid (x, y) \in \mathbb{R}^2, z = T(x, y)\}$$

is called the *tangent plane to the graph of* f *at* (x_0, y_0, z_0).

Equation of the tangent plane $z = T(x, y)$ can be rewritten in the form

$$\begin{aligned} \frac{\partial f}{\partial x}(x_0, y_0) \cdot (x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0) \cdot (y - y_0) - (z - z_0) &= 0 \,, \\ \text{alternatively} \quad \langle \mathbf{n}, (x - x_0, y - y_0, z - z_0) \rangle &= 0 \,, \end{aligned}$$

where $\mathbf{n} \in \mathbb{R}^3$ je vektor

$$\mathbf{n} = \left(\frac{\partial f}{\partial x}(x_0, y_0), \frac{\partial f}{\partial y}(x_0, y_0), -1 \right) .$$

Denoting $\mathbf{x} = (x, y, z)$ and $\mathbf{x_0} = (x_0, y_0, z_0)$, we can express G_T as

$$G_T = \{ \mathbf{x} \in \mathbb{R}^3 \mid \langle \mathbf{n}, \mathbf{x} - \mathbf{x_0} \rangle = 0 \} \,.$$

That is, the tangent plane consists of all points whose direction from $\mathbf{x_0}$ is perpendicular to $\mathbf{n}$. Vector $\mathbf{n}$ is called a *normal vector* of the graph of f at $\mathbf{x_0}$.

Lecture 9 (17.4.2019)

(translated and adapted from lecture notes by Martin Klazar)

Theorem 37 ($\partial \Rightarrow$ differential)**.** *Let $U \subset \mathbb{R}^m$ is a neighborhood of a point $\mathbf{a} \in \mathbb{R}^m$. If a function $f : U \to \mathbb{R}$ has all partial derivatives on U and they are continuous at $\mathbf{a}$, then f is differentiable at $\mathbf{a}$.*

Proof. We consider only the case of two variables x and y ($m = 2$). For more variables, the proof is similar (but more technical). We might assume that the point $\mathbf{a} = \mathbf{o}$ and U is a ball $B(\mathbf{o}, \gamma)$ for some $\gamma > 0$. Let $\mathbf{h} = (h_1, h_2) \in U$ (so, $\|\mathbf{h}\| < \gamma$) and $\mathbf{h}' = (h_1, 0)$. Difference $f(\mathbf{h}) - f(\mathbf{o})$ can be expressed as a sum of differences along both coordinate axes:

$$f(\mathbf{h}) - f(\mathbf{o}) = (f(\mathbf{h}) - f(\mathbf{h}')) + (f(\mathbf{h}') - f(\mathbf{o})) \ .$$

Segments $\mathbf{h}'\mathbf{h}$ and $\mathbf{o}\mathbf{h}'$ lie inside U, so f is defined on them, morever, f depends only on variable y on the former and only on variable x on the latter segment. Thus, Lagrange mean value Theorem (for single variable) yields:

$$f(\mathbf{h}) - f(\mathbf{o}) = \frac{\partial f}{\partial y}(\zeta_\mathbf{2}) \cdot h_2 + \frac{\partial f}{\partial x}(\zeta_\mathbf{1}) \cdot h_1 \ ,$$

where $\zeta_\mathbf{1}$ and $\zeta_\mathbf{2}$ are internal points of segments $\mathbf{o}\mathbf{h}'$ and $\mathbf{h}'\mathbf{h}$, respectively. In particular, the points $\zeta_\mathbf{1}$ and $\zeta_\mathbf{2}$ lie inside $B(\mathbf{o}, \|\mathbf{h}\|)$, so by continuity of both partial derivatives at $\mathbf{o}$, we have

$$\frac{\partial f}{\partial y}(\zeta_\mathbf{2}) = \frac{\partial f}{\partial y}(\mathbf{o}) + \alpha(\zeta_\mathbf{2}) \quad \text{and} \quad \frac{\partial f}{\partial x}(\zeta_\mathbf{1}) = \frac{\partial f}{\partial x}(\mathbf{o}) + \beta(\zeta_\mathbf{1}) \ ,$$

where $\alpha(\mathbf{h})$ and $\beta(\mathbf{h})$ are $o(1)$ as $\mathbf{h} \to \mathbf{o}$ (i.e., for every $\varepsilon > 0$ there is $\delta > 0$, such that $\|\mathbf{h}\| < \delta \Rightarrow |\alpha(\mathbf{h})| < \varepsilon \cdot 1 = \varepsilon$ and the same holds for $\beta(h)$). Thus

$$f(\mathbf{h}) - f(\mathbf{o}) = \frac{\partial f}{\partial y}(\mathbf{o}) \cdot h_2 + \frac{\partial f}{\partial x}(\mathbf{o}) \cdot h_1 + \alpha(\zeta_\mathbf{2})h_2 + \beta(\zeta_\mathbf{1})h_1 \ .$$

By triangle inequality, and inequalities $0 < \|\zeta_\mathbf{1}\|, \|\zeta_\mathbf{2}\| < \|\mathbf{h}\|$ and $|h_1|, |h_2| \leq \|\mathbf{h}\|$ it follows that if $\|\mathbf{h}\| < \delta$, then

$$|\alpha(\zeta_\mathbf{2})h_2 + \beta(\zeta_\mathbf{1})h_1| \leq |\alpha(\zeta_\mathbf{2})| \cdot \|\mathbf{h}\| + |\beta(\zeta_\mathbf{1})| \cdot \|\mathbf{h}\| \leq 2\varepsilon\|\mathbf{h}\| \ .$$

Thus, $\alpha(\zeta_\mathbf{2})h_2 + \beta(\zeta_\mathbf{1})h_1 = o(\|\mathbf{h}\|)$ for $\mathbf{h} \to \mathbf{o}$. So by definition of the total differential, f is differentiable at $\mathbf{o}$. $\square$

Lagrange Mean Value Theorem can be generalized for functions of several variables as follows.

Theorem 38 (Lagrange Mean Value Theorem for several variables)**.** *Let $U \subset \mathbb{R}^m$ be an open set containing a segment $u = \mathbf{ab}$ with endpoints $\mathbf{a}$ and $\mathbf{b}$ and let $f : U \to \mathbb{R}$ be a function which is continuous at every point of u and differentiable at every internal point of u. Then there exists an internal point ζ of u satisfying*

$$f(\mathbf{b}) - f(\mathbf{a}) = \mathrm{D}f(\zeta)(\mathbf{b} - \mathbf{a}) .$$

In other words, difference of functional values at endpoints of the segment equals value of differential at some internal point of the segment for the vector of the segment.

Proof. Idea: Apply Lagrange Mean Value Theorem of single variable for an auxiliary function $F(t) = f(\mathbf{a} + t(\mathbf{b} - \mathbf{a}))$ and $t \in [0, 1]$. □

We say that an open set $D \subset \mathbb{R}^m$ is *connected*, if every two of its points can be connected by a broken line contained in D. Examples of connected open sets: an open ball in $\mathbb{R}^m$, whole $\mathbb{R}^m$ and $\mathbb{R}^3 \backslash L$, where L is the union of finitely many lines. On the other hand, $B \backslash R$, where B is an open ball $\mathbb{R}^3$ and R a plane intersecting B, is an open set which is not connected.

Corollary 39 ($\partial = 0 \Rightarrow f \equiv \text{const.}$)**.** *If a function f of m variables has zero differential at every point of an open connected set U, then f is constant on U. The same conclusion holds if f has all partial derivatives on U zero.*

Proof. Idea: Consider two points of U and a broken line connecting them. Apply Lagrange Mean Value Theorem for several variables for each segment of the broken line.

□

Calculating partial derivatives and differentials. For two functions $f, g : U \to \mathbb{R}$, defined on a neighborhood $U \subset \mathbb{R}^m$ of a point $\mathbf{a} \in U$ that have a partial derivative with repect to x_i at a point $\mathbf{a}$, formulae for partial derivative their sum, product and quotient are analogous to those for single variable:

$$\begin{aligned} \partial_i(\alpha f + \beta g)(\mathbf{a}) &= \alpha \partial_i f(\mathbf{a}) + \beta \partial_i g(a) \\ \partial_i(fg)(a) &= g(\mathbf{a}) \partial_i f(\mathbf{a}) + f(\mathbf{a}) \partial_i g(\mathbf{a}) \\ \partial_i(f/g)(\mathbf{a}) &= \frac{g(\mathbf{a}) \partial_i f(\mathbf{a}) - f(\mathbf{a}) \partial_i g(\mathbf{a})}{g(\mathbf{a})^2} \quad (\text{if } g(\mathbf{a}) \neq 0) . \end{aligned}$$

Similarly, for differentials, we have:

Theorem 40 (Arithmetic of differentials)**.** *Let $U \subset \mathbb{R}^m$ is a neighborhood of $\mathbf{a}$ and $f, g : U \to \mathbb{R}$ are functions differentiable at $\mathbf{a}$.*

(i) $\alpha f + \beta g$ is differentiable at $\mathbf{a}$ and

$$\mathrm{D}(\alpha f + \beta g)(\mathbf{a}) = \alpha \mathrm{D}f(\mathbf{a}) + \beta \mathrm{D}g(\mathbf{a}) .$$

for any $\alpha, \beta \in \mathbb{R}$, *(ii)* fg *is differentiable at* $\mathbf{a}$ *and*

$$\mathrm{D}(fg)(\mathbf{a}) = g(\mathbf{a})\mathrm{D}f(\mathbf{a}) + f(\mathbf{a})\mathrm{D}g(\mathbf{a}) \ .$$

(iii) If $g(\mathbf{a}) \neq 0$, f/g *is differentiable at* $\mathbf{a}$ *and*

$$\mathrm{D}(f/g)(\mathbf{a}) = \frac{1}{g(\mathbf{a})^2}\Big(g(\mathbf{a})\mathrm{D}f(\mathbf{a}) - f(\mathbf{a})\mathrm{D}g(\mathbf{a})\Big) \ .$$

Proof. Follows from Theorem 34 and formulae for partial derivatives. □

The formula for linear combination can be easily generalized for vector valued functions $f, g : U \to \mathbb{R}^n$.

Next, we generalize a formula for derivative of a composed function to a composition of multivariable mappings. We use $\circ$ for denoting composition, where $(g \circ f)(\mathbf{x}) = g(f(\mathbf{x}))$.

Theorem 41 (Differential of a composed mapping)**.** *Let*

$$f : U \to V, \ g : V \to \mathbb{R}^k$$

are two mappings where $U \subset \mathbb{R}^m$ *is a neighborhood of* $\mathbf{a}$ *and* $V \subset \mathbb{R}^n$ *is a neighborhood of* $\mathbf{b} = f(\mathbf{a})$. *If the mapping* f *is differentiable at* $\mathbf{a}$ *and* g *is differentialble at* $\mathbf{b}$, *the composed mapping*

$$g \circ f = g(f) : U \to \mathbb{R}^k$$

is differentiable at $\mathbf{a}$ *and the total differential is a composition of differentials of* f *and* g*:*

$$\mathrm{D}(g \circ f)(\mathbf{a}) = \mathrm{D}g(\mathbf{b}) \circ \mathrm{D}f(\mathbf{a}) \ .$$

Since composition of linear mappings corresponds to multiplication of matrices, total differential of a composed mapping corresponds to a product of the Jacobi matrices.

Lecture 10 (24.4.2019)

(translated and adapted from lecture notes by Martin Klazar)

Partial derivatives of higher orders

If the $f:\ U \to \mathbb{R}$ function defined on a neighborhood $U \subset \mathbb{R}^m$ of a point $\mathbf{a}$ has a partial derivative $F = \partial f x_i$ in each point U and this function $F:\ U \to \mathbb{R}$ has at $\mathbf{a}$ the partial derivative $\partial F x_j(\mathbf{a})$, we say that f has a partial derivative at the point $\mathbf{a}$ of the second order with respect to the variables x_i and x_j and we denote it

$$\frac{\partial^2 f}{\partial x_j \partial x_i}(\mathbf{a})$$

or shortly by $\partial_i \partial_j f(\mathbf{a})$.

Similarly, we define higher order partial derivatives: if $f = f(x_1, x_2,\ ldots, x_m)$ has partial derivative $(i_1, i_2, \ldots, i_{k-1}, j \in \{1, 2, \ldots, m\})$

$$F = \frac{\partial^{k-1} f}{\partial x_{i_{k-1}} \partial x_{i_{k-2}} \ldots \partial x_{i_1}}(x)$$

at every point $x\ inU$ and we say that f has *partial derivative of order k with respect to the variables $x_{i_1}, \ldots, x_{i_{k-1}}, x_j$* in point $\mathbf{a}$ and we denote its value by

$$\frac{\partial^k f}{\partial x_j \partial x_{i_{k-1}} \ldots \partial x_{i_1}}(\mathbf{a})\ .$$

In general, order of variables in higher order derivatives matters. You can verify yourself that $f:\ \mathbb{R}^2 \to \mathbb{R}$,

$$f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2} & \text{pro } x^2+y^2 \neq 0 \\ 0 & \text{pro } x^2+y^2 = 0\ , \end{cases}$$

has different *mixed* (i.e., with respect to two different variables) second order partial derivatives in the origin.

$$\frac{\partial^2 f}{\partial x \partial y}(0,0) = 1 \ \text{ a } \ \frac{\partial^2 f}{\partial y \partial x}(0,0) = -1\ .$$

However, the order does not matter if the partial derivatives are continuous.

Theorem 42 (Usually $\partial_x \partial_y f = \partial_y \partial_x f$)**.** *Let $f:\ U \to \mathbb{R}$ be a function with second order partial derivatives $\partial_j \partial_i f$ a $\partial_i \partial_j f$, $i \neq j$ on a neighborhood $U \subset \mathbb{R}^m$ of a point $\mathbf{a}$ which are continuous in $\mathbf{a}$. Then*

$$\partial_j \partial_i f(\mathbf{a}) = \partial_i \partial_j f(\mathbf{a})\ .$$

Proof. We prove the statement for $m = 2$, for $m > 2$, the proof would be analogous but more tedious. Without loss of generality, we may assume that $\mathbf{a} = \mathbf{o} = (0, 0)$. By continuity of the partial derivatives in the origin, it is enough to find for arbitrarily small $h > 0$ two points σ, τ in the square $[0, h]^2$ satisfying $\partial_x\partial_y f(\sigma) = \partial_y\partial_x f(\tau)$. Then, for $h \to 0^+$, $\sigma, \tau \to \mathbf{o}$ and from a limit argument and continuity of the partial derivatives we get that $\partial_x\partial_y f(\mathbf{o}) = \partial_y\partial_x f(\mathbf{o})$.

Given h, we find σ and τ as follows. We denote the corners of the square $\mathbf{a} = (0, 0)$, $\mathbf{b} = (0, h)$, $\mathbf{c} = (h, 0)$, $\mathbf{d} = (h, h)$ and we consider a value $f(\mathbf{d}) - f(\mathbf{b}) - f(\mathbf{c}) + f(\mathbf{a})$. It can be expressed in two different ways:

$$\begin{aligned} f(\mathbf{d}) - f(\mathbf{b}) - f(\mathbf{c}) + f(\mathbf{a}) &= (f(\mathbf{d}) - f(\mathbf{b})) - (f(\mathbf{c}) - f(\mathbf{a})) = \psi(h) - \psi(0) \\ &= (f(\mathbf{d}) - f(\mathbf{c})) - (f(\mathbf{b}) - f(\mathbf{a})) = \phi(h) - \phi(0) \ , \end{aligned}$$

where

$$\psi(t) = f(h, t) - f(0, t) \text{ and } \phi(t) = f(t, h) - f(t, 0) \ .$$

We have that $\psi'(t) = \partial_y f(h, t) - \partial_y f(0, t)$ and $\phi'(t) = \partial_x f(t, h) - \partial_x f(t, 0)$. Lagrange mean value theorem gives two expresions

$$\begin{aligned} f(\mathbf{d}) - f(\mathbf{b}) - f(\mathbf{c}) + f(\mathbf{a}) &= \psi'(t_0)h = (\partial_y f(h, t_0) - \partial_y f(0, t_0))h \\ &= \phi'(s_0)h = (\partial_x f(s_0, h) - \partial_x f(s_0, 0))h \ , \end{aligned}$$

where $0 < s_0, t_0 < h$ are intermediate points. Applying the theorem once more on differences of partial derivatives of f, we obtain the following

$$f(\mathbf{d}) - f(\mathbf{b}) - f(\mathbf{c}) + f(\mathbf{a}) = \partial_x\partial_y f(s_1, t_0)h^2 = \partial_y\partial_x f(s_0, t_1)h^2, \ s_1, t_1 \in (0, h) \ .$$

Points $\sigma = (s_1, t_0)$ and $\tau = (s_0, t_1)$ belong to $[0, h]^2$ and we have $\partial_x\partial_y f(\sigma) = \partial_y\partial_x f(\tau)$ (since both sides equal to $(f(\mathbf{d}) - f(\mathbf{b}) - f(\mathbf{c}) + f(\mathbf{a}))/h^2$). □

For an open set $U \subset \mathbb{R}^m$ we denote by $\mathcal{C}^k(U)$ the set of functions $f : U \to \mathbb{R}$, such that all their partial derivatives of order up to k (inclusive) (exist and) are continuous on U.

Corollary 43 (Reordering partial derivatives)**.** *For every function $f = f(x_1, x_2, \ldots, x_m)$ from $\mathcal{C}^k(U)$ values of its partial derivatives up to order k do not depend on the order of variables—for $l \le k$ and $\mathbf{a} \in U$ it holds that*

$$\frac{\partial^l f}{\partial x_{i_l}\partial x_{i_{l-1}} \ldots \partial x_{i_1}}(a) = \frac{\partial^l f}{\partial x_{j_l}\partial x_{j_{l-1}} \ldots \partial x_{j_1}}(\mathbf{a}) \ ,$$

whenever $(i_1, \ldots, i_l)$ and $(j_1, \ldots, j_l)$ differ only by permutation of the elements.

Proof. (idea) If a sequence $v = (j_1, \ldots, j_l)$ is a permutation of the sequence $u = (i_1, \ldots, i_l)$, one can turn u into v only by swapping consecutive pairs of elements (in a bubble sort like manner). Then the equality of partial derivatives follows from the previous theorem. □

Since only the multiset of variables matters in case of continuous partial derivatives, we can more briefly write ∂x^2 instead of $\partial_x \partial_x$. For instance, for f from $\mathcal{C}^5(U)$ on U we have

$$\frac{\partial^5 f}{\partial y\, \partial x\, \partial y\, \partial y\, \partial z} = \frac{\partial^5 f}{\partial y^2\, \partial x\, \partial z\, \partial y} = \frac{\partial^5 f}{\partial x\, \partial z\, \partial y^3} = \frac{\partial^5 f}{\partial z\, \partial y^3\, \partial x}\ .$$

Local extrema of multivariate functions

Extrema of the multivariate functions are defined as follows. A function $f:\ U \to \mathbb{R}$, where $U \subset \mathbb{R}^m$ is an open neighborhood of a point $\mathbf{a}$, has in $\mathbf{a}$

- *strict local minimum*, if there exists $\delta > 0$, such that $0 < \|\mathbf{x} - \mathbf{a}\| < \delta \Rightarrow f(\mathbf{x}) > f(\mathbf{a})$,
- *(non-strict) local minimum*, if there exists $\delta > 0$, such that $0 < \|\mathbf{x} - \mathbf{a}\| < \delta \Rightarrow f(\mathbf{x}) \geq f(\mathbf{a})$.

Strict and non-strict local minimum are defined analogously. A function $f:\ M \to \mathbb{R}$, where $M \subset \mathbb{R}^m$, has *maximum on a set* M if $f(\mathbf{a}) \geq f(\mathbf{x})$ for every $\mathbf{x} \in M$. Again, minimum is defined analogously.

Recall facts about extrema of function of a single variable from winter:

1. if $f'(\mathbf{a}) \neq 0$, f does not have a local extremum in $\mathbf{a}$;
2. if $f'(\mathbf{a}) = 0$ and $f''(\mathbf{a}) > 0$, f has a strict local minimum in $\mathbf{a}$ and
3. if $f'(\mathbf{a}) = 0$ and $f''(\mathbf{a}) < 0$, f has a strict local maximum in $\mathbf{a}$.

If $f'(\mathbf{a}) = f''(\mathbf{a}) = 0$, we canot decide whether f has extremum in $\mathbf{a}$ or not without further analysis. If $f'(\mathbf{a}) = 0$ ($\mathbf{a}$ is a "suspicious" point), we cannot, based on the value of the second derivative $f''(\mathbf{a})$ rule out the existence of a local extremum. As we shall see, this is not the case for multivariate functions.

In winter term, it was shown that continuous function has extrema on closed bounded interval. This generalizes to multivariate functions. We say that a set $M \subset \mathbb{R}^m$ is *bounded*, if there exists a real $R > 0$, such that $M \subset B(\overline{0}, R)$. Recall that M is *closed*, if its complement $\mathbb{R}^m \backslash M$ is open. We say that $M \subset \mathbb{R}^m$ is *compact*, when it is closed and bounded.

Theorem 44 (Extrema on compact)**.** *Let $M \subset \mathbb{R}^m$ be a nonempty compact set and $f:\ M \to \mathbb{R}$ a continuous function on M. Then f attains minimum and maximum on M.*

For instance the unit sphere $S = \{\mathbf{x} \in \mathbb{R}^n |\ ||\mathbf{x}|| = 1\}$ in $\mathbb{R}^n$, is a compact set and thus every continuous function $f:\ S \to \mathbb{R}$ attains minimum and maximum on S.

We fist introduce some notation and recall some facts from linear algebra. Let $A = (a_{i,j}) \in \mathbb{R}^{n\times n}$ be a real symmetric matrix ($a_{i,j} = a_{j,i}$) of size $n \times n$. A quadratic form corresponding to this matrix is a function of n variables defined as

$$P_A(x_1, x_2, \dots, x_n) = \mathbf{x}A\mathbf{x}^T = \sum_{i,j=1}^{n} a_{i,j}x_ix_j : \mathbb{R}^n \to \mathbb{R} ,$$

where $\mathbf{x}$ is a row vector $(x_1, x_2, \dots, x_n)$ and $\mathbf{x}^T$ is the corresponding column vector.

A matrix A is

- *pozitive (negative) definite*, if $P_A(\mathbf{x}) > 0$ ($P_A(\mathbf{x}) < 0$) for every $\mathbf{x} \in \mathbb{R}^n \backslash \{\overline{0}\}$;
- *pozitive (negative) semidefinite*, if $P_A(\mathbf{x}) \geq 0$ ($P_A(\mathbf{x}) \leq 0$) for every $\mathbf{x} \in \mathbb{R}^n$ and
- *indefinite*, if it is none of the previous, that is, there exist $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ such that $P_A(\mathbf{x}) > 0$ and $P_A(\mathbf{y}) < 0$.

Hessian matrix $H_f(\mathbf{a})$ of a function f in a point $\mathbf{a}$, where $U \subset \mathbb{R}^m$ is an open neighborhood of $\mathbf{a}$ and $f : U \to \mathbb{R}$ is a function with all partial derivatives of second order on U, is a metrix recording values of these derivatives in $\mathbf{a}$:

$$H_f(a) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{a}) \right)_{i,j=1}^{m} .$$

By theorem that $\partial_x\partial_y = \partial_y\partial_x$, if $f \in \mathcal{C}^2(U)$ its Hessian matrix is symmetric.

Theorem 45 (Necessary condition for local extremum.). *Let $f : U \to \mathbb{R}$, where $U \subset \mathbb{R}^m$ is an open neighborhood of* $\mathbf{a}$. *If $\nabla f(\mathbf{a}) \neq \overline{0}$, then f does not have local extremum in* $\mathbf{a}$.

Proof. For $i = 1, \dots m$, define auxiliary functions of a single variable $g_i(h) = f(\mathbf{a} + h\bar{e}_i)$. Note that $g_i'(0) = \partial f x_i(\mathbf{a})$. By results from winter term, it follows that if $\partial f x_i(\mathbf{a}) \neq 0$, g_i does not have an extremum in 0. Moreover, if g_i does not have an extremum in 0, f does not have an extremum in $\mathbf{a}$. $\square$

Theorem 46 (Sufficient conditions for local extrema). *Let $f \in \mathcal{C}^2(U)$, where $U \subset \mathbb{R}^m$ is an open neighborhood of* $\mathbf{a}$.

1. *If $\nabla f(\mathbf{a}) = \overline{0}$ and $H_f(\mathbf{a})$ is positive (negative) definite, then f has local minimum (maximum) in* $\mathbf{a}$.
2. *If $\nabla f(\mathbf{a}) = \overline{0}$ and $H_f(\mathbf{a})$ is indefinite, f does not have local extremum in* $\mathbf{a}$.

Sylvester kriterion from linear algebra gives the following way to recognize definiteness of a symmetric matrix: if all subdeterminants $d_m = \det(a_{i,j})_{i,j=1}^m$, $1 \leq m \leq n$, are non-zero, then, if all of them are positive, the matrix A is pozitive definite, if $(-1)^m d_m > 0$, $1 \leq m \leq n$, then A is negative definite, and the matrix is indefinite otherwise. (If some of the determinants are zero, we don't know.)

Lecture 11 (15.5.2019)

(translated and slightly adapted from lecture notes by Martin Klazar)

Implicit functions. As we know from linear algebra, system of n linear equations with n variables $a_{i,1}y_1 + a_{i,2}y_2 + \ldots + a_{i,n}y_n + b_i = 0$, $i = 1, 2, \ldots, n$, where $a_{i,j} \in \mathbb{R}$ are given constants and $\det(a_{i,j})_{i,j=1}^n \neq 0$, has for each choice of n constants b_i unique solution $y_1, y_2, \ldots, y_n$. Moreover, this solution y_j is a homogenous linear functions of b_i, that is: $y_j(b_1, b_2, \ldots, b_n) = c_{j,1}b_1 + c_{j,2}b_2 + \ldots + c_{j,n}b_n$, $j = 1, 2, \ldots, n$, for some constants $c_{j,i} \in \mathbb{R}$ (this follows from Crammer's rule).

We now generalize this result to the situation when the linear functions are replaced by general functions. We will consider a system of n equations with $m + n$ variables

$$\begin{aligned} F_1(x_1, \ldots, x_m, y_1, \ldots, y_n) &= 0 \\ F_2(x_1, \ldots, x_m, y_1, \ldots, y_n) &= 0 \\ &\vdots \\ F_n(x_1, \ldots, x_m, y_1, \ldots, y_n) &= 0\ , \end{aligned}$$

where F_i are real functions defined on some neighborhood of a point $(\bar{x}_0, \bar{y}_0)$ in $\mathbb{R}^{m+n}$, where $\bar{x}_0 \in \mathbb{R}^m$ and $\bar{y}_0 \in \mathbb{R}^n$, is a solution of the system, that is $F_1(\mathbf{x}_0, \mathbf{y}_0) = F_2(\mathbf{x}_0, \mathbf{y}_0) = \ldots = F_n(\mathbf{x}_0, \mathbf{y}_0) = 0$. We shall see that under certain conditions it is possible to express variables $y_1, y_2, \ldots, y_n$ as functions $y_i = f_i(x_1, x_2, \ldots, x_m)$ of variables $x_1, x_2, \ldots, x_m$ on some neighborhood of x_0. Even in simplest cases we cannot expect to have necessarily a solution, not to speak of a unique one. Consider example the following single equation

$$F(x, y) = x^2 + y^2 - 1 = 0.$$

For $|x| > 1$ there is no y with $f(x, y) = 0$. For $|x_0| < 1$, we have in a sufficiently small open interval containing x_0 two solutions

$$f(x) = \sqrt{1 - x^2} \text{ and } g(x) = -\sqrt{1 - x^2}.$$

This is better, but we have *two* values in each point, contradicting the definition of a function. To achieve uniqueness, we have to restrict not only the values of x, but *also the values of* y to an interval $(y_0 - \Delta, y_0 + \Delta)$ (where $F(x_0, y_0) = 0$). That is, if we have a particular solution (x_0, y_0) we have a "window"

$$(x_0 - \delta, x_0 + \delta) \times (y_0 - \Delta, y_0 + \Delta)$$

through which we see a unique solution.

But in our example there is also the case $(x_0, y_0) = (1, 0)$, where there is a unique solution, but no suitable window as above, since in every neighborhood of $(1, 0)$, there are no solutions for any value x slightly bigger and two solutions for value x slightly smaller.

Theorem 47 (Implicit function Theorem.). *Let $F(\mathbf{x}, y)$ be a function of $n+1$ variables defined in a neighbourhood of a point $(\mathbf{x}_0, y_0)$. Let F have continuous partial derivatives up to the order $p \geq 1$ and let*

$$F(\mathbf{x}_0, y_0) = 0 \text{ and } \frac{\partial F}{\partial y}(\mathbf{x}_0, y_0) \neq 0.$$

Then there exist $\delta > 0$ and $\Delta > 0$ such that for every $\mathbf{x}$ with $||\mathbf{x} - \mathbf{x}_0|| < \delta$ there exists precisely one y with $|y - y_0| < \Delta$ such that

$$F(\mathbf{x}, y) = 0.$$

Furthermore, if we write $y = f(\mathbf{x})$ for this unique solution y, then the function

$$f : B(\mathbf{x}, \delta) \to \mathbb{R}$$

has continuous partial derivatives up to the order p. Moreover,

$$\frac{\partial f}{\partial x_i}(\mathbf{x}) = -\frac{\frac{\partial F}{\partial x_i}(\mathbf{x}, f(\mathbf{x}))}{\frac{\partial F}{\partial y}(\mathbf{x}, f(\mathbf{x}))}$$

for every $i = 1, \ldots n$.

We will not prove this theorem, however, we show how to derive the formula for partial derivatives of the implicit function f, assuming they exist.

Since we have

$$0 \equiv F(\mathbf{x}, f(\mathbf{x}));$$

taking a derivative of both sides (using the Chain Rule) we obtain.

$$0 = \frac{\partial F}{\partial x_i}(\mathbf{x}, f(\mathbf{x})) + \frac{\partial F}{\partial y}(\mathbf{x}, f(\mathbf{x})) \cdot \frac{\partial f}{\partial x_i}(\mathbf{x}).$$

.

From this, we can express $\frac{\partial f}{\partial x_i}(\mathbf{x})$. Differentiating further, we obtain inductively linear equations from which we can compute the values of all the derivatives guaranteed by the theorem.

For more than a system of several functions, we can apply the previous theorem inductively, eliminating variables one by one.

Theorem 48 (Implicit functions). *Let*

$$F = (F_1, F_2, \ldots, F_n) : \ W \to \mathbb{R}^n$$

be a mapping defined on a neighborhood $W \subset \mathbb{R}^{m+n}$ of a point $(\mathbf{x}_0, \mathbf{y}_0)$, where $\mathbf{x}_0 \in \mathbb{R}^m$ and $\mathbf{y}_0 \in \mathbb{R}^n$, satisfying the following conditions:

1. *$F_i = F_i(\mathbf{x}, \mathbf{y}) \in \mathcal{C}^1(W)$ for $1 \leq i \leq n$.*

2. *$F_i(\mathbf{x}_0, \mathbf{y}_0) = 0$ for $1 \leq i \leq n$.*

3. $\det\left(\begin{pmatrix} \frac{\partial F_1}{\partial y_1} & \frac{\partial F_1}{\partial y_2} & \cdots & \frac{\partial F_1}{\partial y_n} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial F_n}{\partial y_1} & \frac{\partial F_n}{\partial y_2} & \cdots & \frac{\partial F_n}{\partial y_n} \end{pmatrix}(\mathbf{x}_0, \mathbf{y}_0)\right) \neq 0.$

Then there exist neighborhoods $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ of $\mathbf{x}_0$ a $\mathbf{y}_0$ such that $U \times V \subset W$ and for every $\mathbf{x} \in U$ there exists exactly one $\mathbf{y} \in V$ satisfying $F_i(\mathbf{x}, \mathbf{y}) = 0$ for $1 \leq i \leq n$. In other words, there exist a mapping $f = (f_1, f_2, \ldots, f_n) : U \to V$ such that

$$\forall (\mathbf{x}, \mathbf{y}) \in U \times V : \; F(\mathbf{x}, \mathbf{y}) = \overline{0} \iff \mathbf{y} = f(\mathbf{x}) \,.$$

Moreover f_i is $\mathcal{C}^1(U)$ for every $i = 1, \ldots n$.

Constrained extrema. From Implicit functions theorem one can derive a necessary condition for local extrema on sets defined by a system of equations.

Let $U \subset \mathbb{R}^m$ be an open set and let

$$f, F_1, \ldots, F_n : \; U \to \mathbb{R}$$

be functions from $\mathcal{C}^1(U)$, where $n < m$. We wish to find extrema of f on a set

$$H = \{\mathbf{x} \in U \mid F_1(\mathbf{x}) = F_2(\mathbf{x}) = \cdots = F_n(\mathbf{x}) = 0\} \,.$$

Such a set usually does not have any internal points. Example of such a set is a unit sphere in $\mathbb{R}^m$:

$$\{\mathbf{x} \in \mathbb{R}^m \mid x_1^2 + x_2^2 + \ldots + x_m^2 - 1 = 0\} \,.$$

Theorem 49 (Lagrange multipliers)**.** *Let $U \subset \mathbb{R}^m$ be an open set,*

$$f, F_1, \ldots, F_n : \; U \to \mathbb{R}$$

be functions from $\mathcal{C}^1(U)$, where $n < m$ and let

$$H = \{\mathbf{x} \in U \mid F_1(\mathbf{x}) = F_2(\mathbf{x}) = \cdots = F_n(\mathbf{x}) = 0\} \,.$$

Let $\mathbf{a} \in H$. If $\nabla F_1(\mathbf{a}), \ldots, \nabla F_n(\mathbf{a})$ are linearly independent and $\nabla f(\mathbf{a})$ is not their linear combination, then f does not have a local extremum with respect to H in $\mathbf{a}$.

Equivalently: if $\nabla F_1(\mathbf{a}), \ldots, \nabla F_n(\mathbf{a})$ are linearly independent and f has local extremum in $\mathbf{a}$ with respect to H , then there exist reals $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$, called Lagrange multipliers*, such that*

$$\nabla f(\mathbf{a}) - \sum_{i=1}^{n} \lambda_i \nabla F_i(\mathbf{a}) = \overline{0} \,.$$

that is,

$$\frac{\partial f}{\partial x_j}(\mathbf{a}) - \lambda_1 \frac{\partial F_1}{\partial x_j}(\mathbf{a}) - \cdots - \lambda_n \frac{\partial F_n}{\partial x_j}(\mathbf{a}) = 0$$

for every $1 \leq j \leq m$.

Lecture 12 (22.5.2019)

(translated and slightly adapted from lecture notes by Martin Klazar)

Metric and topological spaces

Metric space is a structure formalizing distance. It is a pair (M, d) consisting of $M \neq \emptyset$ and a function of two variables

$$d:\ M \times M \to \mathbb{R},$$

called a *metric*, which satisfies the following three axioms:

- $d(x, y) \geq 0$ (non-negativity) a $d(x, y) = d(y, x)$ (symmetry),
- $d(x, y) = 0 \iff x = y$ and
- $d(x, y) \leq d(x, z) + d(z, y)$ (triangle inequality).

The non-negativity of the metric in a) does not have to be required, it follows from axioms b) and c). Here are some examples of metric spaces. Axioms a) and b) can usually be checked easily. Proving triangle inequality is often more difficult.

Example 5. *$M = \mathbb{R}^n$ a $p \geq 1$ is a real number. At M we define $d_p(x, y)$ metrics*

$$d_p(x, y) = \left(\sum_{i=1}^{n} |x_i - y_i|^p\right)^{1/p}$$

($x = (x_1, x_2, \ldots, x_n), y = (y_1, y_2, \ldots, y_n)$). For $n = 1$ we get classical metrics $|x - y|$ to $\mathbb{R}$ and for $p = 2, n \geq 2$ Euclidean metrics

$$d_2(x, y) = \|x - y\| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}.$$

For $p = 1, n \geq 2$ we get Manhattan metric

$$d_1(x, y) = \sum_{i=1}^{n} |x_i - y_i|$$

and for $p \to \infty$ maximum metric

$$d_\infty(x, y) = \max_{1 \leq i \leq n} |x_i - y_i| \ .$$

Example 6. *For M we take a set of all bounded functions $f:\ X \to \mathbb{R}$ defined on the X set. At M then we have* supremum metric

$$d(f, g) = \sup_{x \in X} |f(x) - g(x)| \ .$$

Example 7. *For a connected graph $G = (M, E)$ with a set of vertices M, we have a metric*

$d(u, v) =$ *the number of edges on the shortest path in G joining vertices u and v*

Example 8. *Let A be a set (alphabet) and let $M = A^m$ be the set of strings of length m over the alphabet A ($u = a_1 a_2 \ldots a_m, v = b_1 b_2 \ldots b_m$). So called* Hamming Metric

$$d(u, v) = \textit{number of coordinates } i\textit{, for which } a_i \neq b_i \ .$$

It measures the degree of difference between the two words, i.e., the smallest number of changes in the letters needed for converting u into v.

We will introduce a few basic concepts; with many we have already met in Euclidean spaces. Let (M, d) be a metric space. Then

- *(open) ball* in M with centre $a \in M$ and radius $\mathbb{R} \ni r > 0$ is the set $B(a, r) = \{x \in M \mid d(a, x) < r\}$;
- $A \subset M$ is *open set* if $\forall a \in A \ \exists r > 0 : \ B(a, r) \subset A$;
- $A \subset M$ is a *closed set* if $M \backslash A$ is an open set;
- $A \subset M$ is a *bounded set* if there is a point $a \in M$ and a radius $r > 0$ that $A \subset B(a, r)$;
- $A \subset M$ is a *compact set* if each sequence of points $(a_n) \subset A$ has a convergent subsequence, whose limit lies in A.

Convergence and limit are generalized from the real axis to the general metric space in an obvious way: sequence $(a_n) \subset M$ is *convergent and has a limit* $a \in M$, (we write $\lim_{n\to\infty} a_n = a$) when

$$\forall \varepsilon > 0 \ \exists n_0 : \ n > n_0 \Rightarrow d(a_n, a) < \varepsilon$$

In other words, $\lim_{n\to\infty} d(a_n, a) = 0$ (we have converted it to the real sequence limit).

We have already mentioned the properties of open sets: $\emptyset$ and M are open, union of any system sets of open sets is an open set, and the intersection of any finite system of open sets is an open set. By switching to the complement, we have the dual properties of closed sets: $\emptyset$ and M are closed, the union of any finite system of closed sets is a closed set, and the intersection of any set system of closed sets is a closed set.

Theorem 50 (Characterisation of closed sets)**.** *A set $A \subset M$ is closed in M, if and only if the limit of every convergent sequence $(a_n) \subset A$ belongs to A.*

Proof. Let $A \subset M$ be a closed set and $(a_n) \subset A$ a convergent sequence. If $\lim_{n\to\infty} a_n = a \notin A$, there exists a radius $r > 0$ such that $B(a,r) \subset M \backslash A$. But then $d(a_n, a) \geq r$ for every n, this contradicts that $\lim_{n\to\infty} a_n = a$. So $a \in A$.

Conversely, if the $A \subset M$ is not a closed set, there is a point $a \in M \backslash A$ such that for each radius $r > 0$ is $B(a,r) \cap A \neq \emptyset$. We put $r = 1/n$, $n = 1, 2, \ldots$, and for each n choose a point $a_n \in B(a, 1/n) \cap A$. Then $(a_n) \subset A$ is a convergent sequence with $\lim_{n\to\infty} a_n = a$, but $a \notin A$. □

Topological spaces. Topological spaces are generalization of metric spaces. The pair $T = (X, \mathcal{T})$, where X is the set and $\mathcal{T}$ is a system of its subsets is a topological space if $\mathcal{T}$ has the following properties:

(i) $\emptyset, X \in \mathcal{T}$,

(ii) $\bigcup \mathcal{U} \in \mathcal{T}$ for every subsystem $\mathcal{U} \subset \mathcal{T}$, and

(iii) $\bigcap \mathcal{U} \in \mathcal{T}$ for every finite subsystem $\mathcal{U} \subset \mathcal{T}$.

Sets in the $\mathcal{T}$ system is called the *open sets* of the topological space T (their complements to X are then *closed sets* of the T space). Example of topological space are the open sets of each metric space. However, there are plenty of topological spaces, which are not metrizable (i.e. do not come from metric space).

Continuous mappings. Let (M, d) and (N, e) be two metric spaces. We say that a mapping

$$f: M \to N$$

is *continuous*, if

$$\forall a \in M, \varepsilon > 0 \; \exists \delta > 0 : \; b \in M, d(a,b) < \delta \Rightarrow e(f(a), f(b)) < \varepsilon \; .$$

.

Theorem 51 (Topological definition of continuity). *A mapping $f: M \to N$ between metric spaces is continuous, if and only if for every open set $B \subset N$ is its preimage $f^{-1}(B) = \{x \in M \mid f(x) \in B\}$ open set in M.*

Theorem 52 (Compact $\Rightarrow$ closed and bounded). *Each compact set in the metric space is closed and bounded.*

Proof. Let $A \subset M$ be a subset in metric space (M, d). When A is not closed, there is convergence the sequence $(a_n) \subset A$, whose limit a does not belong to A. Each subsequence of (a_n) is also convergent and has the same limit a. This means that no subsequence (a_n) is has its limit within A (the limit is determined unambiguously) and thus A is not compact.

When A is not bounded, it is not contained in any $B(a,r)$ balls and we can easily build a sequence $(a_n) \subset A$ with the property that $d(a_m, a_n) \geq 1$ for every two indices $1 \leq m < n$. This property contradicts sequence convergence (why?) every subsequence of (a_n)has this property , so (a_n) has no convergent subsequence. A is not compact again.

We define a sequence $(a_n) \subset A$ with the specified property inductively. We take the first point $a_1 \in A$ arbitrarily. Assume that we have already constructed points $a_1, a_2, \ldots, a_k$ of A, such that the distance of each pair is at least 1. Then we take any sphere $B(a, r)$, which contains all of these points (each finite set is bounded) and consider the $B(a, r+1)$ sphere. Since A is not bounded, there exists point $a_{k+1} \in A$ that is not in $B(a, r+1)$. According to the triangle inequality, $d(a_{k+1}, x) \geq 1$ for every point $x \in B(a, r)$ (why?). Thus a_{k+1} has distance at least 1 from each point $a_1, a_2, \ldots, a_k$ a $a_1, a_2, \ldots, a_k$ we can extend to $a_1, a_2, \ldots, a_k, a_{k+1}$. Thus defined sequence a_k, $k = 1, 2, \ldots$ has the required property. $\square$

Probably the simplest example showing that the converse does not hold in general is the following. Let (M, d) be a trivial metric space, where $d(x, y) = 1$ for $x \neq y$ a $d(x, x) = 0$ (verify that this is a metric space), and the M set is infinite. Then each the sequence $(a_n) \subset M$, where a_n are mutually different points (for the existence of such a sequence we need infinity M) satisfies that $d(a_m, a_n) \geq 1$ for every two indices $1 \leq m < n$. As we know, such a sequence has no convergent subsequence and therefore M is not a compact set. But M is a closed set and it is also bounded because it is a subset of $B(a, 2)$ for any point $a \in M$.

As we have already mentioned, the converse holds for the Euclidean spaces.

Theorem 53 (Closed and bounded $\Rightarrow$ compact in $\mathbb{R}^k$). *Each closed and bounded set in the Euclidean space $\mathbb{R}^k$ is compact.*

Theorem 54 (Continuous function attains extremes on compact). *Let $f : M \to \mathbb{R}$ be a continuous function from the metric space (M, d) into the Euclidean space $\mathbb{R}^1$ and M is compact. Then f has minimum and maximum on M.*