Lecture 3 (5.3.2018)

(translated and slightly adapted from lecture notes by Martin Klazar)

Riemann integral

Now we define precisely the concept of the area, in particular, the area of figure $U(a,b,f)$ under the graph of a function f. Let $-\infty < a < b < +\infty$ be two real numbers and $f:\ [a,b] \to \mathbb{R}$ any function that may not be continuous or bounded. The finite $k+1$-tuple of points $D = (a_0, a_1, \ldots, a_k)$ from the interval $[a,b]$ is called a *partition* of $[a,b]$ if

$$a = a_0 < a_1 < a_2 < \ldots < a_k = b\ .$$

These points divide the interval $[a,b]$ into intervals $I_i = [a_{i-1}, a_i]$. We denote by $|I_i|$ the length of interval I_i: $|I_i| = a_i - a_{i-1}$ a $|[a,b]| = b - a$. Clearly

$$\sum_{i=1}^{k} |I_i| = (a_1 - a_0) + (a_2 - a_1) + \ldots + (a_k - a_{k-1}) = b - a = |[a,b]|\ .$$

Norm of a partition D is the maximum length of an interval of the partition and is denoted by λ:

$$\lambda = \lambda(D) = \max_{1 \le i \le k} |I_i|\ .$$

Partition of an interval $[a,b]$ *with points* is a pair (C,D) wher $D = (a_0, a_1, \ldots, a_k)$ is a partition of $[a,b]$ and a k-tuple $C = (c_1, c_2, \ldots, c_k)$ consists of $c_i \in I_i$ (i.e. $a_{i-1} \le c_i \le a_i$). *Riemann sum* corresponding to the function f and a partition with points (D,C) is defined as

$$R(f,D,C) = \sum_{i=1}^{k} |I_i| f(c_i) = \sum_{i=1}^{k} (a_i - a_{i-1}) f(c_i)\ .$$

If $f \ge 0$ on $[a,b]$, it is the sum of k rectangles $I_i \times [0, f(c_i)]$ whose union approximates figure $U(a,b,f)$. However, Riemann sum is defined for every function f, regardless of its sign on $[a,b]$. The following definition was introduced by Bernhard Riemann (1826–1866).

Definition (first definition of Riemann integral, Riemann). *We say that* $f:[a,b] \to \mathbb{R}$ *has Riemann integral* $I \in \mathbb{R}$ *on* $[a,b]$ *if for every* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that for each partition of* $[a,b]$ *with points* (D,C) *such that* $\lambda(D) < \delta$ *the following holds:*

$$|I - R(f,D,C)| < \varepsilon$$

Therefore, we require $I \in \mathbb{R}$, values $\pm\infty$ are not allowed (although, it is possible to define them). If there is such a number I, we write

$$I = \int_a^b f(x)\, dx = \int_a^b f = (\mathcal{R}) \int_a^b f$$

and say that f is *Riemann integrable* on the interval $[a, b]$. We will work with the class of all Riemann integrable functions

$$\mathcal{R}(a, b) := \{f \mid f \text{ is defined and Riemann integrable on } [a, b]\}.$$

Thus, the first definition of the Riemann integral can be summarized by the formula

$$\int_a^b f = \lim_{\lambda(D)\to 0} R(f, D, C) \in \mathbb{R}\,.$$

We understand the limit here as defined in the definition above; as a symbol, we defined only limit of a sequence and of a function in a point.

For the second, equivalent, definition of the integral we will need a few more concepts. For $f : [a, b] \to \mathbb{R}$ and a partition $D = (a_0, a_1, \ldots, a_k)$ of interval $[a, b]$ we define *lower and upper Riemann sum*, respectively, (even though they were introduced by Darboux) as

$$s(f, D) = \sum_{i=1}^{k} |I_i| m_i, \text{ and } S(f, D) = \sum_{i=1}^{k} |I_i| M_i\,,$$

where

$$m_i = \inf_{x\in I_i} f(x) \text{ and } M_i = \sup_{x\in I_i} f(x)$$

$$I_i = [a_{i-1}, a_i]$$

These sums are always defined $s(f, D) \in \mathbb{R} \cup \{-\infty\}$ and $S(f, D) \in \mathbb{R} \cup \{+\infty\}$ *Lower and upper Riemann integral*, respectively, of a function f on the interval $[a, b]$ is defined as

$$\underline{\int_a^b} f = \underline{\int_a^b} f(x)\, dx = \sup(\{s(f, D) :\ D \text{ is a partition of } [a, b]\})\,,$$

and

$$\overline{\int_a^b} f = \overline{\int_a^b} f(x)\, dx = \inf(\{S(f, D) :\ D \text{ is a partition of } [a, b]\})\,.$$

These terms are always defined and we have $\underline{\int_a^b} f, \overline{\int_a^b} f \in \mathbb{R}^* = \mathbb{R} \cup \{-\infty, +\infty\}$.

Definition (second definition of Riemann integral, Darboux). *We say that $f : [a, b] \to \mathbb{R}$ has at $[a, b]$ Riemann integral, if*

$$\underline{\int_a^b} f(x)\, dx = \overline{\int_a^b} f(x)\, dx \in \mathbb{R}\,.$$

This common value, if it exists, we denote by

$$\int_a^b f(x)\,dx = \int_a^b f$$

and we call it the Riemann integral of f on the interval $[a,b]$.

The two definitions are equivalent: they give the same classes of Riemann integrable functions and the same value of the Riemann integral, if defined.

Claim (unbounded functions have no integral). *If the $f:\ [a,b] \to \mathbb{R}$ function is not bounded then it does not have a Riemann integral on $[a,b]$, according to both definitions.*

When $D = (a_0, a_1, \ldots, a_k)$ a $D' = (b_0, b_1, \ldots, b_l)$ are partitions of $[a,b]$ and $D \subset D'$, that is for every $i = 0, 1, \ldots, k$ there exists j, such that $a_i = b_j$ (therefore $k \le l$), we say that D' *is a refinement of* D or that D' *refines* D.

Lemma. *If $f:\ [a,b] \to \mathbb{R}$ and D, D' are two partitions of $[a,b]$, and D' refines D,*

$$s(f,D') \ge s(f,D) \ \textit{and}\ \ S(f,D') \le S(f,D).$$

Proof. Considering the definition of $s(f,D)$ a $S(f,D)$ and the fact that D' can be created from D by adding points, it is enough to prove both inequalities in a situation where $D = (a_0 = a < a_1 = b)$ a $D' = (a'_0 = a < a'_1 < a'_2 = b)$. According to the definition of infima f, we have

$$m_0 = \inf_{a_0 \le x \le a_1} f(x) \le \inf_{a'_0 \le x \le a'_1} f(x) = m_0 \inf_{a'_1 \le x \le a'_2} f(x) = m'_1$$

Then

$$\begin{aligned} s(f,D') &= (a'_1 - a'_0)m'_0 + (a'_2 - a'_1)m'_1 \\ &\ge (a'_1 - a'_0)m_0 + (a'_2 - a'_1)m_0 \\ &= (a'_2 - a'_0)m_0 = (b-a)m_0 \\ &= s(f,D)\ . \end{aligned}$$

Proof of the inequality $S(f,D') \le S(f,D)$ is similar. □

Corollary. *When $f:\ [a,b] \to \mathbb{R}$ and D, D' are two partitions $[a,b]$, then*

$$s(f,D) \le S(f,D')\ .$$

Proof. Let $E = D \cup D'$ be a common refinement of both partitions. According to the previous lemma we have

$$s(f,D) \le s(f,E) \le S(f,E) \le S(f,D')$$

More precisely, the first and last inequality follow from the previous lemma, and the middle one from the definition of upper and lower sum. □

Theorem (lower integral does not exceed upper). *Let $f:\ [a,b] \to \mathbb{R}$, $m = \inf_{a\le x\le b} f(x)$, $M = \sup_{a\le x\le b} f(x)$ and D, D' be two partitions of interval $[a,b]$. Then the following inequalities hold:*

$$m(b-a) \le s(f,D) \le \underline{\int_a^b} f \le \overline{\int_a^b} f \le S(f,D') \le M(b-a)\ .$$

Proof. The first and last inequality are the special cases of the previous lemma. The second and penultimate inequality comes straight from the definition of the lower and upper integral as supremum or infimum respectively. According to the corollary, each element is a set of lower sums whose supremum is $\underline{\int_a^b} f$ smaller or equal to each element of the upper sum set whose infim is $\overline{\int_a^b} f$. Using the definition of infimum (the largest lower bound) and supremum (the smallest upper bound) we get the middle inequality: For each partition D, $s(f,D)$ the lower bound of the set of upper sums, that is, $s(f,D) \le \overline{\int_a^b} f$, and so $\overline{\int_a^b} f$ is the upper bound of the set of lower sums, thus $\underline{\int_a^b} f \le \overline{\int_a^b} f$. □

Theorem (Integrability criterion). *Let $f:\ [a,b] \to \mathbb{R}$. Then*

$$f \in \mathcal{R}(a,b) \iff \forall\varepsilon > 0\ \exists D:\ 0 \le S(f,D) - s(f,D) < \varepsilon.$$

In other words, f has Riemann's integral if and only if for every $\varepsilon > 0$ there exists a partition of D of interval $[a,b]$ such that its upper Riemann sum is greater than the corresponding lower Riemann sum by less than ε.

Proof. " $\Rightarrow$ " We assume that f has R. integral on $[a,b]$, i.e., $\underline{\int_a^b} f = \overline{\int_a^b} f = \int_a^b f \in \mathbb{R}$. Let $\varepsilon > 0$ be given. By definition of the lower and upper integrals, there are partitions E_1 and E_2 so that

$$s(f,E_1) > \underline{\int_a^b} f - \frac{\varepsilon}{2} = \int_a^b f - \frac{\varepsilon}{2} \ \text{ a }\ S(f,E_2) < \overline{\int_a^b} f + \frac{\varepsilon}{2} = \int_a^b f + \frac{\varepsilon}{2}\ .$$

According to the lemma, these inequalities also apply after replacing E_1 and E_2 with their joint refinement $D = E_1 \cup E_2$. Summing up both inequalities we will get

$$S(f,D) - s(f,D) < \int_a^b f + \frac{\varepsilon}{2} + \Big(- \int_a^b f + \frac{\varepsilon}{2}\Big) = \varepsilon\ .$$

" $\Leftarrow$ " Given $\varepsilon > 0$ we take a partition of D satisfying the condition. According to the definition of the lower and upper integral we get

$$\overline{\int_a^b} f \le S(f,D) < s(f,D) + \varepsilon \le \underline{\int_a^b} f + \varepsilon, \text{ thus } \overline{\int_a^b} f - \underline{\int_a^b} f < \varepsilon\ .$$

This inequality is valid for every $\varepsilon > 0$, so according to the previous statement we have $\overline{\int_a^b} f = \underline{\int_a^b} f \in \mathbb{R}$. Then f has R. integral on $[a,b]$. □

Example (bounded function without integral). *A function* $f:\ [0,1] \to \{0,1\}$ *defined as* $f(\alpha) = 1$ *when* α *is a rational number, and* $f(\alpha) = 0$*, when* α *is irrational, is called* Dirichlet function*, and does not have Riemann integral on* $[0,1]$*, although bounded.*

Each positive-length interval contains points where f has a value of 0, as well as points that have a value of 1. Then $s(f,D) = 0$ and $S(f,D) = 1$ for every partition of D and therefore

$$\underline{\int_0^1} f = 0 < \overline{\int_0^1} f = 1\ .$$