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- Lovász Local Lemma: Generalization when A_i are only partially dependent.

Dependency (di)graph

Definition

We say that an event A is independent of events $B_1, \ldots, B_k$ if for every $J \subseteq [k], J \neq \emptyset$ we get:

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Let $A_1, \ldots, A_n$ be events. A directed graph $D = (V, E)$ with $V = \{1, \ldots, n\}$ is a dependency (di)graph for $A_1, \ldots, A_n$ if for every A_i is independent of the events A_j with $(i, j) \notin E$.

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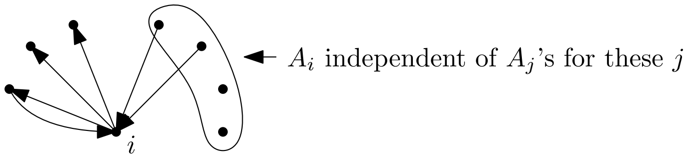

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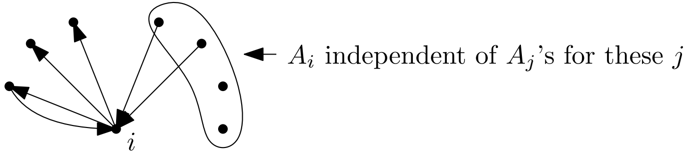

- Dependency graph is not unique in general.

Symmetric Lovász local lemma

Theorem (Symmetric Lovász local lemma)

Let $A_1, \ldots, A_n$ be events such that $\forall i\colon P[A_i] \leq p$ where $p \in (0,1)$. Assume also that the outdegrees in a dependency graph are at most d. (That is, $\forall i\colon$ A_i is independent on all but at most d events.) If $ep(d+1) \leq 1$, then

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Standard probabilistic technique

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Let $A_1, \ldots, A_n$ be events and $D = (V, E)$ be their dependency graph. Let $x_i \in [0, 1)$ be such that $P[A_i] \leq x_i \prod_{(i,j) \in E} (1 - x_j)$. Then $P\left[\bigcap_{i=1}^{n} \bar{A}_i\right] \geq \prod_{i=1}^{n} (1 - x_i) > 0$.

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- In concrete applications, it is often useful to set $x_i \sim c \cdot P[A_i]$.

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  $\ge \frac{1}{e(d+1)} \ge p \ge P[A_i]$. $\square$

Applications: Coloring hypergraphs

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Let $H = (V, E)$ be a hypergraph in which each hyperedge contains at least k vertices and it meets at most d other hyperedges. If $e(d+1) \leq 2^{k-1}$, then H is 2-colorable. (No edge monochromatic.)

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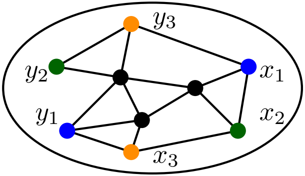
- Comparison to union bound: $P[\bigcap \bar{A}_f] > 0$ if $\sum P[A_f] < 1$, that is, if $|E| < 2^{k-1}$. (Depends on the number of edges.)

Applications: Edge disjoint paths

- Given a graph G and pairs $\{x_i, y_i\} \subseteq V(G)$.
- Target: connect each x_i with y_i with mutually edge disjoint paths.



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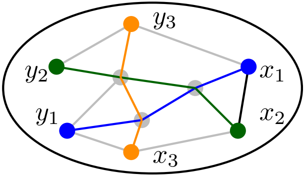

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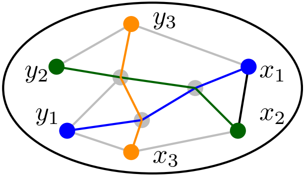

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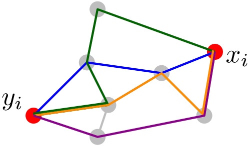

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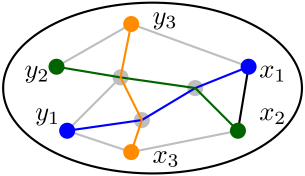

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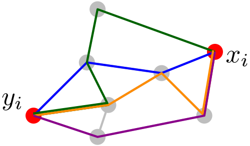

- If Q_i are large enough and the paths from Q_i do not share edges with too many paths from Q_j, then the x_i–y_i connections can be simultaneously established.

Edge disjoint paths–continued

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- By symmetric Lovász local lemma, $P[\bigcap \bar{A}_{i,j}] > 0$. $\square$

Directed cycles via LLL

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Let $D = (V, E)$ be a directed graph with minimum outdegree δ and maximum indegree Δ. Then for every positive integer k satisfying $k \leq \frac{\delta}{1+\ln(1+\delta\Delta)}$ there is a directed cycle in D of length divisible by k.

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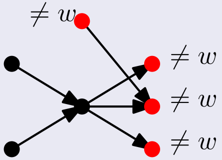

Directed cycles via LLL–continued

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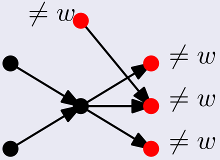

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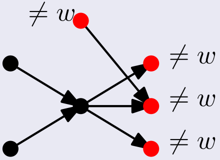

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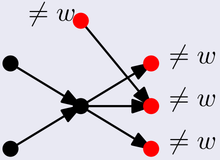

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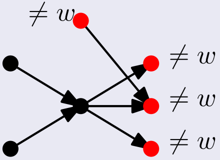

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- Then $ep(d+1) \leq e(1-1/k)^\delta(\delta\Delta + 1) \leq e^{1-\frac{\delta}{k}}(\delta\Delta+1) \leq 1$.
- By Lovász local lemma, there is a coloring such that $\forall v \exists w \in N^+(v)$ colored with $f(v)+1$.
- Start with v_0 and gradually build the cycle. $\square$