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- We would also like to get $P[G(n,p)$ contains a triangle$] \to 1$ if $p = \omega(1/n)$.
- Not obvious: If F is the number of Hungarian forints owned by a random person in the world. Then $E[F] \geq 1000$ does not imply that almost everybody owns at least a single forint.

Threshold functions

Definition

Graph property A is monotone if whenever G and H are graphs with $V(G) = V(H)$ and $E(G) \supseteq E(H)$, then if H admits property A, then G admits property A as well.

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A function $r\colon \mathbb{N} \to \mathbb{R}$ is a threshold function (prahová funkce) for a monotone graph property A, if for every $p\colon \mathbb{N} \to [0, 1]$ we have:

- If $p(n) = o(r(n))$, then $\lim_{n\to\infty} P[G(n, p) \text{ admits } A] = 0$.
- If $r(n) = o(p(n))(\Leftrightarrow p(n) = \omega(r(n)))$, then $\lim_{n\to\infty} P[G(n, p) \text{ admits } A] = 1$.

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$r(n) = \frac{1}{n}$ is a threshold function for triangle containment.

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- We already checked the first condition; the second one remains.

Auxiliary lemma

Lemma

Let $X_1, X_2, \ldots,$ *be a sequence of non-negative random variables such that*

$$\lim_{n\to\infty} \frac{\mathrm{Var}[X_n]}{(E[X_n])^2} = 0.$$

Then $\lim_{n\to\infty} P[X_n > 0] = 1$.

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Proof.

- $P[|X_n - E[X_n]| \geq E[X_n]] \leq \frac{\mathrm{Var}[X_n]}{(E[X_n])^2}$ by Chebyshev's inequality.

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- $P[|X_n - E[X_n]| \geq E[X_n]] \leq \frac{\mathrm{Var}[X_n]}{(E[X_n])^2}$ by Chebyshev's inequality.
- Then $\lim_{n\to\infty} P[X_n \leq 0] \leq \lim_{n\to\infty} \frac{\mathrm{Var}[X_n]}{(E[X_n])^2} = 0$. □

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- $\mathrm{Var}[T_i] = E[T_i^2] - (E[T_i])^2 \leq E[T_i^2] = E[T_i] = p^3$.

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- $\text{Cov}[T_i, T_j] \begin{cases} \leq E[T_i T_j] = p^5 \text{ if } T_i \text{ and } T_j \text{ share an edge} \\ = 0 \text{ if } T_i T_j \text{ do not share an edge.} \end{cases}$

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- $\text{Var}[T] \leq \binom{n}{3} p^3 + \binom{n}{2}(n-2)(n-3)p^5 \leq n^3 p^3 + n^4 p^5$.

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- Then

$$\frac{\mathrm{Var}[T]}{(E[T])^2} \leq \frac{n^3 p^3 + n^4 p^5}{\binom{n}{3}^2 p^6} = O\left(\frac{1}{n^3 p^3} + \frac{1}{n^2 p}\right) \overset{p=\omega(1/n)}{\rightarrow} 0.$$

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- By auxiliary lemma: $P[G(n,p)$ contans a triangle$] \to 1$.

□

Generalization to balanced graphs

Definition

Let H be a graph with v vertices and e edges. Then the density of H is defined as $\rho(H) = \frac{e}{v}$. We say that H is balanced if $\rho(H') \leq \rho(H)$ for an arbitrary subgraph H' of H.

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Generalization to balanced graphs—proof

Theorem

Let H be a balanced graph with density ρ. Then $r(n) = n^{-1/\rho}$ is a threshold function for the property: H is a subgraph of $G(n, p)$.

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- For a v-tuple $\beta = (b_1, \ldots, b_v)$ of distinct vertices in $G(n,p)$ let A_β be the event expressing that $b_i b_j \in E(G(n,p))$ whenever $a_i, a_j \in E(H)$. (In other words $G(n,p)$ contains a copy of H in the given order.)

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- Let X_β be the indicator of A_β; $X := \sum_\beta X_\beta$.
- $X \neq$ no. of copies of H but $X > 0 \Leftrightarrow G(n,p)$ contains H.

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- If $p(n) = o(n^{-v/e})$, then $\lim_{n\to\infty} E[X] = 0$. Therefore $\lim_{n\to\infty} P[X > 0] = 0$ as needed.

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Proof—continued.

- Remains $p(n) = \omega(n^{-v/e})$.

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- Thus $\mathrm{Cov}[X_\beta, X_\gamma] \leq E[X_\beta X_\gamma] \leq p^{2e-\rho t}$.

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- Thus $\mathrm{Cov}[X_\beta, X_\gamma] \leq E[X_\beta X_\gamma] \leq p^{2e-\rho t}$.
- Number of pairs (β, γ) sharing $t \geq 2$ vertices is $O(n^{2v-t})$: There $\binom{n}{2v-t}$ options how to pick $\beta \cup \gamma$, then there is only constant number of options how to determine H.

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- $\sum_{|\beta\cap\gamma|=t} \text{Cov}[X_\beta, X_\gamma] = O(n^{2v-\rho t} p^{2e-t}) = O((n^v p^e)^{2-t/v})$.

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- $\sum_{|\beta\cap\gamma|=t} \mathrm{Cov}[X_\beta, X_\gamma] = O(n^{2v-\rho t} p^{2e-t}) = O((n^v p^e)^{2-t/v})$.
- Thus $\mathrm{Var}[X] = O\left(\sum_{t=2}^{v} (n^v p^e)^{2-t/v}\right)$.

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- Recall $p(n) = \omega(n^{-\frac{v}{e}})$, $E[X] = \Theta(n^v p^e)$, $\mathrm{Var}[X] = O\big(\sum_{t=2}^{v} (n^v p^e)^{2-\frac{t}{v}}\big)$.

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- $\lim_{n\to\infty} \frac{\mathrm{Var}[X]}{(E[X])^2} = \lim_{n\to\infty} \big(\sum_{t=2}^{v} (n^v p^e)^{-t/v}\big) = 0$

Generalization to balanced graphs—proof, continued

Proof, continued.

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- By Auxiliary lemma: $\lim_{n\to\infty} P[X > 0] = 1$; that is, $G(n, p)$ contains a copy of H with probability tending to 1 as n tends to ∞. □